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Matrix model eigenvalue integrals and twist fields in the $su(2)$-WZW model

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Abstract: We propose a formula for the eigenvalue integral of the hermitian one matrix model with infinite well potential in terms of dressed twist fields of the su(2) level one WZW model. The expression holds for arbitrary matrix size n, and provides a suggestive interpretation for the monodromy properties of the matrix model correlators at finite n, as well as in the $1/n$-expansion.

Keywords: Conformal and W Symmetry, Matrix Models, 1/N Expansion, 2D Gravity.

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Contents

1. Introduction

It has been known for some time that there is a close relation between the hermitian one matrix model and the conformal field theory of one free boson [36, 29, 42, 33]. One of the quantities that appears naturally in the matrix model context corresponds, in conformal field theory, to the operator

$$I_\lambda(a,b) = \exp\left(\frac{\lambda}{2\pi}\int_a^b J^+(x)\,dx\right). \tag{1.1}$$

Here λ is a complex number and we have expressed the free boson theory (at the self-dual radius) in terms of the su(2) level one WZW model with the standard basis of the weight one fields denoted by $J^+(z)$, $J^3(z)$ and $J^-(z)$. Since the operator product expansion of $J^+(z)J^+(w)$ is regular, no normal ordering is required to ensure convergence of the integrals that occur upon expanding (1.1).

As will be explained in more detail in the next subsection, the matrix model analysis suggests that the operator $I_\lambda(a,b)$ will exhibit interesting monodromy properties for the su(2) fields in the limit of large matrix size n. The aim of this paper is to elucidate these in terms of the above conformal field theory description. In particular, we shall propose that the operator (1.1) can also be expressed in terms of dressed twist fields (see section 1.2 below for more details) which will make these monodromy properties manifest. To leading order in $1/n$ the relation between $I_\lambda(a,b)$ and twist fields had been suggested before in [33, 13]; here we shall propose an exact relation for all n.

1.1 Relation to matrix models and 2d gravity

The matrix integral for the hermitian one matrix model is given by

$$Z_{\rm mm}\left[\underline{t}\right]^{(n)} = ({\rm const})\int d\Phi\, e^{-\frac{1}{g_s}{\rm tr}\,W(\Phi)} = \int_{-\infty}^{\infty} d\lambda_1\ldots d\lambda_n \prod_{i,j=1,\,i<j}^{n}(\lambda_i-\lambda_j)^2 e^{-\frac{1}{g_s}\sum_{k=1}^n W(\lambda_k)}\,. \tag{1.2}$$

The first integral is over all hermitian $n\times n$ matrices, while the second integral amounts to expressing the first one in terms of the eigenvalues of Φ, and $W(x) = \sum_{m\geq 0} t_m x^m$ is the potential; more details can be found e.g. in the review [11]. The relation to the free boson conformal field theory is established by noting that the integrand of (1.2) can be written as a correlator of free boson vertex operators [36, 29, 42, 33]. Using the language of su$(2)_1$ the expression is,

$$\langle n|e^{-H}J^+(x_1)\cdots J^+(x_n)|0\rangle = e^{-\frac{1}{g_s}\sum_{k=1}^n W(x_k)}\prod_{i<j}(x_i-x_j)^2\,, \tag{1.3}$$

where $H = \frac{1}{g_s}\int_{\mathcal{C}_\infty} W(z)J^3(z)\frac{dz}{2\pi i}$. This formula is most easily verified by employing the operator product expansion $J^3(z)J^+(w) = (z-w)^{-1}J^+(w) + O(1)$ to commute e^{-H} to the right. Here we have assumed that $W(x)$ is analytic on $\mathbb{C}$: this allows us to use contour integrals, and guarantees that the only contributions come from the J^+-insertions. Using charge conservation, the matrix integral can then be expressed in terms of the exponential (1.1) as

$$\langle n|e^{-H}I_\lambda(-\infty,\infty)|0\rangle = \frac{\lambda^n}{(2\pi)^n\,n!}Z_{\rm mm}\left[\underline{t}\right]^{(n)}. \tag{1.4}$$

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The correspondence extends also to correlators. Let $Z_{\rm mm}[\underline{t}, f]^{(n)}$ stand for the the right hand side of (1.2), with an additional factor f in the integrand. For the matrix model resolvent we then have the relation

$$\langle n|e^{-H} J^3(z) I_\lambda(-\infty,\infty)|0\rangle = \frac{\lambda^n}{(2\pi)^n\, n!} Z_{\rm mm}\left[\underline{t}, \operatorname{tr}\frac{1}{z-M} - \frac{1}{2g_s}W'(z)\right]^{(n)}, \qquad (1.5)$$

as can be verified by writing out the integrals explicitly and comparing the integrands.

Provided the potential $W(x)$ increases fast enough for $x \to \pm\infty$, the eigenvalues of the matrix Φ will condense, in the large-n limit, on one or more intervals on the real axis, the so-called cuts (for details, consult e.g. [11]). The correlator $Z_{\rm mm}\left[\underline{t}, \operatorname{tr}\frac{1}{z-M}\right]^{(n)}$ has, again in the large-n limit, square root branch cuts on these intervals. Equivalently, continuing $J^3(z)$ in (1.5) around an endpoint of a cut in the large-n limit results in $J^3 \to -J^3$. This motivates the idea to model the endpoints of matrix model cuts by insertions of twist fields $\sigma(x)$ on the conformal field theory side [33, 13]. Free boson twist fields have also been considered in the context of the integrable hierarchy approach to two-dimensional gravity in [16, 20, 21].

We will investigate this effect in a simplified setting, in which we choose $W(x)$ in (1.2) to be an infinite well potential, that is, $W(x) = 0$ for $x \in [a,b]$ and $W(x) = +\infty$ otherwise. This effectively restricts the eigenvalue integrations from the real axis to the interval $[a,b]$, so that the relation (1.4) takes the simpler form

$$\langle n|I_\lambda(a,b)|0\rangle = \frac{\lambda^n}{(2\pi)^n\, n!} Z_{\rm mm}^{\rm well,(n)}\,. \qquad (1.6)$$

Matrix integration measures (or potentials) which force the eigenvalues to lie on a contour which has an endpoint on the complex plane, rather then to start and end at infinity, are referred to as ensembles with hard edges, see e.g. [6, 18] where such models are treated and more references can be found. From the conformal field theory point of view, the relation (1.6) is easier to analyse than (1.4) because the potential term e^{-H} is absent. For example, for the one-point functions one finds

$$\begin{aligned} \langle n|J^3(z) I_\lambda(a,b)|0\rangle &= \frac{\lambda^n}{(2\pi)^n\, n!} Z_{\rm mm}^{\rm well}\left[\operatorname{tr}\frac{1}{z-M}\right]^{(n)} \\ \langle n{\pm}1|J^\pm(z) I_\lambda(a,b)|0\rangle &= \frac{\lambda^n}{(2\pi)^n\, n!} Z_{\rm mm}^{\rm well}\left[\det(z-M)^{\pm 2}\right]^{(n)}\,. \end{aligned} \qquad (1.7)$$

It should be stressed that even if we choose the interval $[a,b]$ to coincide with the location of a cut in a matrix model with analytic potential $W(x)$, the large-n expansion of the free energies

$$\ln\left(\langle n|e^{-H} J^3(z) I_\lambda(a,b)|0\rangle\right) \qquad \text{and} \qquad \ln\left(\langle n|e^{-H} J^3(z) I_\lambda(-\infty,\infty)|0\rangle\right), \qquad (1.8)$$

as well as their behaviour in the double scaling limit, will be different. The reason is that for subleading effects in n^{-1}, the decay-behaviour of the eigenvalues just outside of the cut will be important, and that has precisely been cut off by the infinite well potential[1].

[1]We thank B. Eynard for a discussion on this point.

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As we have explained above one may expect, given the monodromy properties of the matrix model correlators, that the operator $I_\lambda(a,b)$ is proportional to the product of two twist fields to leading order in $1/n$. For finite n, the monodromy properties are however quite different. Indeed, the correlator $Z_{\rm mm}[\underline{t}, \det(z-M)]^{(n)}$ is single valued for finite n (being a polynomial of degree n), but has jumps at the location of the cuts in the $1/n$-expansion. This is an example of Stokes' phenomenon (for an exposition see for example [3], [34, app. B]): the analytic continuation in z of an asymptotic expansion (the $1/n$-expansion in this case) of a function $f(n,z)$ can be different from the asymptotic expansion of the analytic continuation.

The double scaling limit of the hermitian matrix model, with an appropriately tuned potential, describes a $(p,2)$-minimal model coupled to Liouville gravity (see e.g. [11]) or, equivalently, $(p,2)$-minimal string theory (see [43] for a summary of recent developments). The above effect has been given an interesting target space interpretation in the context of minimal string theory [34]. There, the target space is identified with the moduli space of FZZT branes and it is shown that while in the semi-classical limit this moduli space becomes a branched covering of the complex plane, in the exact quantum description the moduli space is in fact much simpler, being just the complex plane itself. On the matrix model side, the analogue of the string partition function in the presence of a FZZT brane is the correlator of the exponentiated macroscopic loop operator, $Z_{\rm mm}[\underline{t}, \det(z-M)]^{(n)}$ [2, 40, 41, 30]. As described above, this correlator is single valued for finite n but develops branch cuts in the large-n limit. This is an example for how classical geometry emerges as an effective concept in string theory.

As we will see, also the simplified matrix model with the infinite well potential that we consider in this paper exhibits this behaviour. The alternative formula for $I_\lambda(a,b)$ in terms of dressed twist fields that we shall propose will then give a suggestive explanation of this phenomenon.

1.2 Summary of results

To explain more precisely our formula for $I_\lambda(a,b)$ in terms of dressed twist fields, we first need to give a few definitions, starting with the relevant twist fields $\sigma_{\pm\lambda}(z)$. Around an insertion of $\sigma_{\pm\lambda}(z)$, the three su(2)-currents have a $\mathbb{Z}_2$-monodromy

$$(J^+, J^3, J^-) \longmapsto (\lambda^{-2}J^-, -J^3, \lambda^2 J^+)\,. \tag{1.9}$$

If instead of J^c one uses the basis

$$K^3 = \frac{1}{2}\big(\lambda J^+ + \lambda^{-1}J^-\big), \qquad K^\nu = \frac{\nu}{2}\big(\lambda J^+ - \lambda^{-1}J^-\big) - J^3 \qquad \text{for } \nu = \pm 1\,, \tag{1.10}$$

the twist fields $\sigma_{\pm\lambda}(z)$ have the standard monodromy $K^3 \mapsto K^3$ and $K^\nu \mapsto -K^\nu$. In particular, the field $K^3(z)$ is single valued, and the two twist fields $\sigma_{+\lambda}(z)$ and $\sigma_{-\lambda}(z)$ are distinguished by their K^3-charge (i.e. the eigenvalue of the K^3 zero mode), which is $\frac{1}{4}$ and $-\frac{1}{4}$, respectively. In the K-basis it is obvious that the monodromy around $\sigma_{\pm\lambda}(z)$ amounts to an inner automorphism of su(2) of order two.

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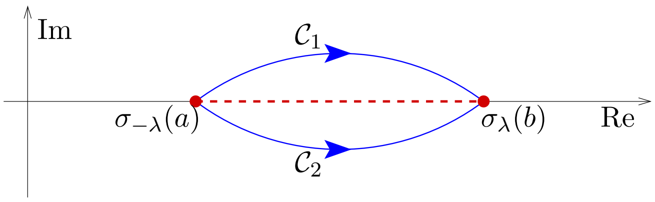

Figure 1: The contours in the definition of $S_\lambda(a,b)$. The dashed line represents the branch cut between the two twist fields.

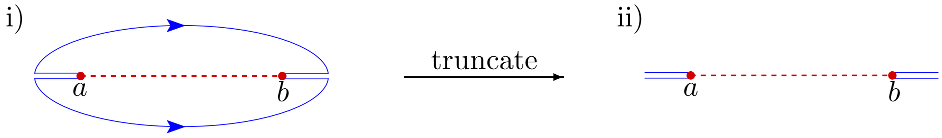

Figure 2: Deforming and truncating the integration contours.

Let us define an operator $S_\lambda(a,b)$ which is very similar in spirit to (a special case of) the star-operators introduced in [38, 39]. Explicitly, it is given in terms of twist fields and exponentiated J^--integrals as follows,

$$S_\lambda(a,b) = (b-a)^{\frac{1}{8}} \left[\sigma_{+\lambda}(b) \exp\left(-\frac{1}{2\pi\lambda}\int_{\mathcal{C}_1} J^-(x)dx\right) \exp\left(-\frac{1}{2\pi\lambda}\int_{\mathcal{C}_2} J^-(x)dx\right) \sigma_{-\lambda}(a)\right]_{\text{reg}} . \tag{1.11}$$

Here $\mathcal{C}_1$ is an integration contour from a to b passing above the interval $[a,b]$, the contour $\mathcal{C}_2$ has the same endpoints, but passes below the interval, and $[\cdots]_{\text{reg}}$ refers to a prescription to regulate the first order pole in the operator product expansion of J^- and $\sigma_{\pm\lambda}$ (see sections 5.2 and 5.4 below). The contours are illustrated in figure 1.

As we shall explain below in section 2.3, the product of the two twist fields $\sigma_{+\lambda}(b)\sigma_{-\lambda}(a)$ that appears in $S_\lambda(a,b)$ can be expressed in terms of an exponentiated integral of the form

$$\sigma_{+\lambda}(b)\sigma_{-\lambda}(a) = (b-a)^{-\frac{1}{8}} \; : \exp\left(\frac{1}{4}\int_a^b \left(\lambda\, J^+(z) + \lambda^{-1} J^-(z)\right)\right) : \, . \tag{1.12}$$

Qualitatively speaking, the J^- integrals in the formula (1.11) for $S_\lambda(a,b)$ can be interpreted as removing the J^- part of this integral, leaving behind only the J^+ integrals that appear in $I_\lambda(a,b)$. This observation motivates the following operator identity for the two rather different looking exponentiated integrals (1.1) and (1.11),

$$I_\lambda(a,b) = S_\lambda(a,b)\, . \tag{1.13}$$

This equality is the main result of our paper. We have no complete proof for it; the supporting evidence will be given in section 5.

As a consequence of (1.13) it is now possible to see that the operator $I_\lambda(a,b)$ does indeed display the monodromy properties described in the previous section. Consider a correlator of the form $\langle n|(\text{fields})S_\lambda(a,b)|0\rangle$, where (fields) stands for any product of the currents $J^c(z)$. The integration contour of figure 1 can be deformed as in figure 2i). It turns out (see section 5.3) that the part of the integral along the ellipse is suppressed by a factor r^{-2n} for some $r > 0$. This results in the approximation

$$\langle n|(\text{fields})S_\lambda(a,b)|0\rangle = \langle n|(\text{fields})S_\lambda^{\text{trunc}}(a,b)|0\rangle \left(1 + O\left(r^{-2n}\right)\right), \tag{1.14}$$

where $S_\lambda^{\text{trunc}}(a,b)$ is defined as $S_\lambda(a,b)$, but with the J^--integrals taken only over the short horizontal contours shown in figure 2 ii).[2] The difference between $S_\lambda(a,b)$ and $S_\lambda^{\text{trunc}}(a,b)$ in a correlator with an out-state of charge n is thus non-perturbative in $1/n$. In particular, both correlators in (1.14) will have the same $1/n$ expansion (the correlators have to be normalised appropriately to allow a $1/n$-expansion, see section 5.3). But since in $S_\lambda^{\text{trunc}}(a,b)$ the points a and b are no longer connected by J^- integrals, the monodromy of the currents $J^c(z)$ around the points a and b is just the $\mathbb{Z}_2$-monodromy (1.9) of the twist fields. On the other hand, the monodromy of $I_\lambda(a,b)$ (and hence that of $S_\lambda(a,b)$) is not given by (1.9). For example, in the presence of $I_\lambda(a,b)$ the current $J^+(z)$ is single valued (since the operator product expansion $J^+(z)J^+(w)$ is regular), while under (1.9) it changes to $\lambda^{-2}J^-(z)$. In this sense, the 'correct' monodromy of the currents $J^c(z)$ in the presence of $I_\lambda(a,b)$ is a non-perturbative effect and cannot be seen in the $1/n$-expansion. This is the same effect as observed in the previous section for the minimal string, albeit here in a different model, namely a matrix integral with hard edges.

Further, in the large-n limit itself, the J^--integrals are suppressed altogether, and $S_\lambda(a,b)$ can be replaced by a product of twist fields. Taking into account the need to regulate (1.11) (see section 5.3), we arrive in this way at the second important result of our paper,

$$\langle n|(\text{fields})S_\lambda(a,b)|0\rangle = n^{-\frac{1}{4}}2^{\frac{1}{12}}e^{3\zeta'(-1)}(b-a)^{\frac{1}{8}}\langle n|(\text{fields})\sigma_{+\lambda}(b)\sigma_{-\lambda}(a)|0\rangle \left(1 + O\left(n^{-1}\right)\right), \tag{1.15}$$

where ζ is the Riemann zeta-function. This shows that our formula for $S_\lambda(a,b)$ also has the correct large-n limit.

In passing from $I_\lambda(a,b)$ to $S_\lambda(a,b)$ we have effectively decomposed the monodromy across the interval $[a,b]$ into a product of three terms, one each associated with one of the lines in figure 1 (the explicit product can be found in (5.2) below). This is analogous to a method introduced in [10] to analyse Riemann-Hilbert problems, which can also be applied to investigate the asymptotics of orthogonal polynomials (see [4, 9], and the lecture notes [8]). There, the orthogonal polynomials are encoded in a Riemann-Hilbert problem with an appropriate jump matrix across the real line, and their large-n behaviour can be found by manipulating the contour along which the jump condition is imposed in a way analogous to figure 1.

The paper is organised as follows. In section 2 we review how free boson vertex operators and twist fields can be expressed as exponentials of integrated currents. To compare the properties of $I_\lambda(a,b)$ and twist fields, we calculate some correlators of su$(2)_1$-currents in the presence of two twist fields (section 3) and in the presence of $I_\lambda(a,b)$

[2] As explained in section 5.3, in addition the regulator introduces an overall factor.

(section 4). Finally, the definition of $S_\lambda(a,b)$ is given in section 5, where also its properties are investigated. Section 6 contains our conclusions. We have also included two appendices where some of the more technical calculations are described.

2. Representing fields as exponentiated integrals

Let us begin by explaining how one can represent fields in terms of exponentiated integrals. While this may seem unfamiliar at first, there is at least one example where this construction is actually well known. This is the case of a free boson that we shall review first.

2.1 The case of the u(1) representation

The free boson theory with field $X(z,\bar{z})$ has an u(1) symmetry that is generated by a weight one current $H(z) = i\partial X(z,\bar{z})$ with operator product expansion

$$H(z)\,H(w) = \frac{1}{2(z-w)^2} + O(1)\,. \tag{2.1}$$

In terms of modes, $H(z)$ can be expanded as $H(z) = \sum_n H_n z^{-n-1}$. These modes then satisfy the commutation relations

$$[H_m, H_n] = \frac{1}{2}\, m\delta_{m,-n}\,. \tag{2.2}$$

The corresponding stress energy tensor is $T(z) = \,:\!H(z)H(z)\!:$, where the colons denote normal ordering, i.e.

$$:H(z)H(z): \; = \lim_{w\to z}\left(H(w)H(z) - \frac{1}{2(w-z)^2}\right). \tag{2.3}$$

The modes of the stress energy tensor $T(z) = \sum_n L_n z^{-n-2}$ define a Virasoro algebra with $c = 1$; in terms of the modes H_n we have

$$L_n = \sum_m : H_m\, H_{n-m} :, \tag{2.4}$$

where the colons denote here the usual normal ordering of modes.

An (untwisted) highest weight representations of the u(1) theory is generated by a state $|\mu\rangle$ that is annihilated by the modes H_n with $n > 0$, and is an eigenvector of H_0 with eigenvalue μ,

$$H_n\,|\mu\rangle = \mu\delta_{n,0}\,|\mu\rangle\,, \qquad n \geq 0\,. \tag{2.5}$$

The corresponding vertex operator will be denoted by V_μ, and can be described by the usual vertex operator construction

$$V_\mu(z) = \,:e^{2i\mu X(z)}:. \tag{2.6}$$

We would now like to express this operator in terms of the current $H(z)$ of the conformal field theory. At least formally we can write $iX(z) = \int^z H(w)dw$, and thus we should be able to write the vertex operator V_μ in terms of an exponentiated integral. However, the exponentiated integral will have a non-vanishing vacuum expectation value, and thus it will not just describe the field V_μ, but rather the pair of V_μ together with its conjugate $V_{-\mu}$. Thus one is led to expect [25]

$$V_\mu(b)\, V_{-\mu}(a) = (b-a)^{-2\mu^2} \; : \exp\left(2\mu \int_a^b H(z)\, dz\right) : , \tag{2.7}$$

where the prefactor is needed to produce the correct scaling behaviour, as will be discussed further below (see (2.10)). In fact, one can show that this identity holds in arbitrary correlation functions. (For a definition of conformal field theory in terms of correlation functions see for example [25].) To this end one observes that the $V_{\pm\mu}$ satisfy indeed their defining relations (2.5) since one calculates, using Wick's Theorem,

$$\begin{aligned}
&\Big\langle \prod_{i=1}^n H(u_i)\, H(w) \; : \exp\left(2\mu \int_a^b H(z)\, dz\right) : \Big\rangle = \\
&\qquad = \mu \left(\frac{1}{w-b} - \frac{1}{w-a}\right) \Big\langle \prod_{i=1}^n H(u_i) \; : \exp\left(2\mu \int_a^b H(z)\, dz\right) : \Big\rangle + \\
&\qquad\quad + \frac{1}{2} \sum_{j=1}^n \frac{1}{(w-u_j)^2} \Big\langle \prod_{i\neq j} H(u_i) : \exp\left(2\mu \int_a^b H(z)\, dz\right) : \Big\rangle,
\end{aligned} \tag{2.8}$$

where we have used that

$$\int_a^b \frac{dz}{(w-z)^2} = \frac{1}{w-b} - \frac{1}{w-a} \,. \tag{2.9}$$

Since the u_i are arbitrary, it follows that any correlation function of $H(w)$ with the integrated exponential has only poles in $(w-a)$ and $(w-b)$ of order one; the fields at a and b are therefore highest weight states. It is also manifest from the above formula (by taking the contour integral around a or b) that their eigenvalues with respect to H_0 are $\pm\mu$.

The above analysis determines the exponential up to an overall function of $b-a$. This is fixed by considering the vacuum expectation value of (2.7), which equals

$$\langle V_\mu(b)\, V_{-\mu}(a)\rangle = (b-a)^{-2\mu^2} \,. \tag{2.10}$$

This is of the form $(b-a)^{-2h}$ (as required by conformal symmetry) precisely for the choice of prefactor made in (2.7). It is easy to see, by the same methods as above and using the definition of the stress energy tensor in terms of the u(1) field, that the fields that are defined by the right hand side of (2.7) also have the correct conformal weight.

As a non-trivial consistency check, one can confirm that these highest weight fields then also give rise to the correct 4-point functions. To this end we consider

$$\begin{aligned}
&\big\langle V_\mu(b_1)\, V_{-\mu}(a_1)\, V_\nu(b_2)\, V_{-\nu}(a_2)\big\rangle = \\
&\quad = (b_1-a_1)^{-2\mu^2} (b_2-a_2)^{-2\nu^2} \Big\langle : \exp\left(2\mu \int_{a_1}^{b_1} H(z)\, dz\right) :: \exp\left(2\nu \int_{a_2}^{b_2} H(w)\, dw\right) : \Big\rangle.
\end{aligned} \tag{2.11}$$

The correlator on the right hand side can now be easily evaluated and one obtains

$$
\begin{aligned}
(2.11) &= (b_1-a_1)^{-2\mu^2}(b_2-a_2)^{-2\nu^2}\sum_{l=0}^{\infty}\frac{2^l\mu^l\nu^l}{l!l!}\sum_{\sigma\in S_l}\left(\prod_{i=1}^{l}\int_{a_1}^{b_1}dz_i\int_{a_2}^{b_2}dw_i\right)\prod_{j=1}^{l}\frac{1}{(z_j-w_{\sigma(j)})^2}\\
&= (b_1-a_1)^{-2\mu^2}(b_2-a_2)^{-2\nu^2}\sum_{l=0}^{\infty}\frac{2^l\mu^l\nu^l}{l!}\prod_{i=1}^{l}\int_{a_1}^{b_1}dz_i\left(\frac{1}{z_i-b_2}-\frac{1}{z_i-a_2}\right)\\
&= (b_1-a_1)^{-2\mu^2}(b_2-a_2)^{-2\nu^2}\left(\frac{(b_1-b_2)(a_1-a_2)}{(a_1-b_2)(b_1-a_2)}\right)^{2\mu\nu}. \qquad (2.12)
\end{aligned}
$$

This then agrees with the known answer.

2.2 Representations of su(2)$_1$

At the self-dual radius the free boson theory is actually equivalent to an su(2) current theory with $k=1$. The su(2) current symmetry is generated by three currents $J^{\pm}$ and J^3 of conformal weight one with operator product expansion[3]

$$
\begin{aligned}
J^+(z)J^-(w) &= \frac{k}{(z-w)^2}+\frac{2J^3(w)}{z-w}+O(1)\\
J^3(z)J^{\pm}(w) &= \pm\frac{J^{\pm}(w)}{z-w}+O(1)\\
J^3(z)J^3(w) &= \frac{k}{2(z-w)^2}+O(1)\,. \qquad (2.13)
\end{aligned}
$$

The modes of J^3 and $J^{\pm}$ then satisfy the commutation relations

$$
\begin{aligned}
[J^+_m,J^-_n] &= 2\,J^3_{m+n}+k\,m\,\delta_{m,-n}\\
[J^3_m,J^{\pm}_n] &= \pm\,J^{\pm}_{m+n}\\
[J^3_m,J^3_n] &= \frac{k}{2}\,m\,\delta_{m,-n}\,. \qquad (2.14)
\end{aligned}
$$

At level $k=1$, the su(2) theory has only two irreducible representations: the vacuum representation, and the representation with $j=\frac{1}{2}$. Here j denotes the spin of the su(2)$_k$ highest weight representation whose states of lowest conformal weight are labelled by $|j,m\rangle$ with $m=-j,-j+1,\ldots,j-1,j$. For $j=\frac{1}{2}$ there are only two highest weight states, namely $|\frac{1}{2},\frac{1}{2}\rangle$ and $|\frac{1}{2},-\frac{1}{2}\rangle$ that are conjugate to one another. It is easy to see that the corresponding vertex operators $V(|j,m\rangle,z)$ can then be obtained by the previous construction provided we take $\mu=\pm\frac{1}{2}$

$$
V\left(\left|\frac{1}{2},\frac{1}{2}\right\rangle,b\right)V\left(\left|\frac{1}{2},-\frac{1}{2}\right\rangle,a\right)=(b-a)^{-\frac{1}{2}}:\exp\left(\int_a^b J^3(z)\,dz\right):\,. \qquad (2.15)
$$

For larger values of $k\geq 2$, however, the above construction does not account for all non-trivial representations. In fact, in addition to the vacuum representation (that corresponds to $\mu=0$) only the highest weight state $|\frac{k}{2},\frac{k}{2}\rangle$ together with its conjugate state $|\frac{k}{2},-\frac{k}{2}\rangle$ can be described in the above manner (with $\mu=\frac{1}{2}$).

[3]For $k=1$ this then agrees with the above u(1) theory by setting $J^3=H$ and $J^{\pm}=V_{\pm 1}$.

2.3 Twisted representations of the $\mathrm{su}(2)_1$ theory

Up to now we have only discussed the untwisted highest weight representations of su(2). The corresponding vertex operators have the property that the currents J^3 and $J^\pm$ are *single-valued* around the insertion point of the vertex operator. As is well known, the su(2) theory (like any affine theory) has also *twisted representations* for which the currents have non-trivial monodromies around the insertion points of the vertex operators. For su(2) all of these twisted representations are actually equivalent (as affine representations) to untwisted representation, since all automorphisms of SU(2) are inner. However since the relevant identification modifies the energy momentum tensor, twisted representations describe often different physical systems (for an introduction to these matters see for example [26]).

In the following we shall mainly be interested in $\mathbb{Z}_2$-twisted representations of $\mathrm{su}(2)_1$. One class of $\mathbb{Z}_2$-twisted representations have the property that the monodromy of the currents is described by the (inner) automorphism

$$J^3 \mapsto J^3 , \qquad J^\pm \mapsto -J^\pm . \tag{2.16}$$

The corresponding representation then has modes J^3_n that are integer valued ($n \in \mathbb{Z}$), while the modes $J^\pm_r$ are half-integer valued ($r \in \mathbb{Z} + \frac{1}{2}$); these modes then still satisfy the same commutation relations (2.14) as above. Since these twisted representations are in one-to-one correspondence with the usual untwisted representations, there are two inequivalent irreducible $\mathbb{Z}_2$-twisted representations for $\mathrm{su}(2)_1$: they are generated from highest weight states $\tilde\sigma_\pm$ with $J^3_0\tilde\sigma_\pm = \pm\frac{1}{4}\tilde\sigma_\pm$. From what was explained above, it is then clear that these vertex operators can also be described by exponentiated integrals; indeed we have simply

$$\tilde\sigma_+(b)\tilde\sigma_-(a) = (b-a)^{-\frac{1}{8}} : \exp\left(\frac{1}{2}\int_a^b J^3(z)\, dz\right) : . \tag{2.17}$$

In the following another class of $\mathbb{Z}_2$-twisted representations will play an important role: these are the $\mathbb{Z}_2$-twisted representations for which the monodromy is described by[4]

$$J^3 \mapsto -J^3 , \qquad J^\pm \mapsto J^\mp . \tag{2.18}$$

Obviously, this only differs by a field redefinition from (2.16): if we define (*cf.* (1.10) with $\lambda = 1$)

$$K^3 = \frac{1}{2}\big(J^+ + J^-\big) , \qquad K^\pm = \pm\frac{1}{2}\big(J^+ - J^-\big) - J^3 , \tag{2.19}$$

then the fields $K^\pm$ and K^3 satisfy the same operator product expansion as $J^\pm$ and J^3 (and thus their modes have the same commutation relations as (2.14)). Furthermore, the monodromy (2.18) has the same form as (2.16). In particular, the two highest weight states $\sigma_\pm$ that are now characterised by the condition that $K^3_0\sigma_\pm = \pm\frac{1}{4}\sigma_\pm$ are given by

$$\sigma_+(b)\sigma_-(a) = (b-a)^{-\frac{1}{8}} : \exp\Big(\frac{1}{4}\int_a^b \big(J^+(z) + J^-(z)\big)dz\Big) : . \tag{2.20}$$

This is the formula that will motivate our ansatz for $S_\lambda(a,b)$ (see section 5.1).

[4] see (1.9) with $\lambda = 1$. The case of general λ differs from $\lambda = 1$ by the rescaling $J^\pm \mapsto \lambda^{\pm 1} J^\pm$. For the following it is therefore sufficient to consider $\lambda = 1$.

3. Properties of the su(2)-twist fields

As explained in the introduction, we propose that the operator $I_\lambda(a,b)$ given in (1.1) can be expressed in terms of twist fields as in (1.11). To support this claim, we shall compare, in section 5, a number of properties of $I_\lambda(a,b)$ and $S_\lambda(a,b)$. As a preparation, we now want to study the correlation functions of the su(2) twist fields.

3.1 Monodromy and zero-point function

We will start with the zero-point function $\langle n|\sigma_+(b)\sigma_-(a)|0\rangle$. Here $\langle n|$ denotes an out-state which is highest weight with respect to J^3 and has J^3-charge n,

$$\langle n|\, J^3_m = n\,\delta_{m,0}\langle n| \qquad \text{for } \ m \le 0\,. \tag{3.1}$$

For $n \ge 0$ one can write $\langle n|$ explicitly in terms of J^- modes (for $n \le 0$ one has to use J^+)

$$\langle n| = \langle 0|J^-_1 J^-_3 \cdots J^-_{2n-1}\,. \tag{3.2}$$

Let us normalise $\langle 0|0\rangle = 1$. One verifies that with the above definition $\langle n|n\rangle = 1$, as well as

$$\langle n|J^-_m = 0 \quad \text{for } m \le 2n \qquad \text{and} \qquad \langle n|J^+_m = 0 \quad \text{for } m \le -2n\,, \tag{3.3}$$

where we have used the null-vector relations of $\mathrm{su}(2)_1$. The highest weight property $\langle n|J^3_m = 0$ for $m < 0$ and $\langle n|J^3_0 = n\langle n|$ are then immediate consequences of the commutation relations (2.14). In order to evaluate $\langle n|\sigma_+(b)\sigma_-(a)|0\rangle$ one can write the out-state in the form (3.2) and express the J^- modes in terms of the K-basis (2.19). For example, abbreviating $|\sigma\sigma\rangle \equiv \sigma_+(b)\sigma_-(a)|0\rangle$,

$$\langle 1|\sigma\sigma\rangle = \langle 0|J^-_1|\sigma\sigma\rangle = \Big\langle 0\Big| \Big(K^3_1 - \frac{1}{2}(K^+_1 - K^-_1)\Big)\Big|\sigma\sigma\Big\rangle\,. \tag{3.4}$$

As is clear from the discussion in section 2.3, the field K^3 has only a simple pole with $\sigma_\pm$,

$$K^3(z)\sigma_\pm(a) = \pm\frac{1}{4}\frac{1}{(z-a)} + O(1)\,, \tag{3.5}$$

and thus it is easy to evaluate the term involving K^3_1, giving $\langle 0|K^3_1|\sigma\sigma\rangle = \frac{1}{4}(b-a)\langle 0|\sigma\sigma\rangle$. For $K^\pm$ one can write (*cf.* also [23])

$$\langle 0|K^\pm_1|\sigma\sigma\rangle = \int_{\mathcal{C}_\infty} z\langle 0|K^\pm(z)|\sigma\sigma\rangle\frac{dz}{2\pi i} = \int_{\mathcal{C}_\infty} \sqrt{(z-a)(z-b)}\langle 0|K^\pm(z)|\sigma\sigma\rangle\frac{dz}{2\pi i} = 0\,. \tag{3.6}$$

To see the second equality, expand the square root in z^{-1} and use the highest weight property of $\langle 0|$. In the third step the contour $\mathcal{C}_\infty$ is deformed around the insertions $\sigma_+(b)$ and $\sigma_-(a)$ and the highest weight property of the latter is used.

Altogether we thus obtain $\langle 1|\sigma\sigma\rangle = \frac{1}{4}(b-a)\langle 0|\sigma\sigma\rangle$. For higher values of n the calculation is similar, but more tedious. Luckily, the exact answer is known from factorising the correlator of four twist fields [44, 45], [17, eq. (4.17)],

$$\langle n|\sigma_+(b)\sigma_-(a)|0\rangle = 4^{-n^2}(b-a)^{n^2-\frac{1}{8}}\,, \tag{3.7}$$

where we normalised the twist fields such that $\langle 0|\sigma\sigma\rangle = 1\cdot(b-a)^{-\frac{1}{8}}$.

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3.2 Correlators involving su(2)-currents

The next correlator we consider is the one-point function of $J^3(z)$ in the presence of two twist fields. It is easily determined by considering the function

$$f(z) = \sqrt{(z-a)(z-b)}\langle n|\, J^3(z)\sigma_+(b)\sigma_-(a)|0\rangle \tag{3.8}$$

and noting that $f(z)$ is single valued on the complex plane and does not have any poles. Since $\langle n|J^3_0 = n\langle n|$ we furthermore have $\lim_{z\to\infty} f(z) = n\,\langle n|\,\sigma_+(b)\sigma_-(a)\,|0\rangle$ so that $f(z)$ is in fact a constant. In this way we find

$$\langle n|\, J^3(z)\sigma_+(b)\sigma_-(a)\,|0\rangle = \frac{n}{\sqrt{(z-a)(z-b)}}\langle n|\sigma_+(b)\sigma_-(a)\,|0\rangle\,. \tag{3.9}$$

Correlators with several J^3 insertions can be determined in a similar fashion.

Finally we will need correlators with several $J^\pm$ insertions in the presence of two twist fields. These can be determined from the Knizhnik-Zamolodchikov equation, that is, by solving the first order differential equations resulting from the null-vector

$$(L_{-1} - 2\nu J^3_{-1})|J^\nu\rangle = 0\,, \qquad \text{where} \qquad \nu\in\{\pm 1\}\,. \tag{3.10}$$

Knizhnik-Zamolodchikov equations in the presence of twist fields have been studied in [5]. To find the differential equations first note that as a consequence of this null-vector we have the identity

$$\int_{\mathcal{C}_z} \frac{\sqrt{(u-a)(u-b)}}{u-z}\, J^3(u)J^\nu(z)\frac{du}{2\pi i} = \frac{\nu}{2}\left(\frac{2z-a-b}{\sqrt{(z-a)(z-b)}} + \sqrt{(z-a)(z-b)}\frac{\partial}{\partial z}\right)J^\nu(z)\,, \tag{3.11}$$

where $\mathcal{C}_z$ is a contour winding closely around the point z. Here we have used the operator product expansion

$$J^3(u)\,J^\nu(z) = \frac{\nu}{(u-z)}J^\nu(z) + (J^3_{-1}J^\nu)(z) + O(u-z)\,. \tag{3.12}$$

Around a point $w\neq z$ and around infinity we find analogously

$$\begin{aligned}\int_{\mathcal{C}_w} \frac{\sqrt{(u-a)(u-b)}}{u-z} J^3(u)J^\nu(w)\frac{du}{2\pi i} &= \nu\frac{\sqrt{(w-a)(w-b)}}{w-z}J^\nu(w)\\ \int_{\mathcal{C}_\infty} \frac{\sqrt{(u-a)(u-b)}}{u-z}\langle n|J^3(u)\frac{du}{2\pi i} &= n\langle n|\,. \end{aligned}\tag{3.13}$$

Consider now the integral

$$\int_{\mathcal{C}_{z_k}} \frac{\sqrt{(u-a)(u-b)}}{u-z_k}\langle n|J^3(u)J^{\nu_1}(z_1)\cdots J^{\nu_m}(z_m)|\sigma\sigma\rangle\frac{du}{2\pi i}\,. \tag{3.14}$$

This contour integral can be calculated in two ways: on the one hand, we can directly use (3.11) and thus evaluate the contour integral in terms of the right hand side of (3.11). On the other hand, we can deform the contour around z_k to encircle all other insertion points z_i, $i \neq k$ as well as infinity and a and b. As in the previous calculation, there is no contribution from the twist-field insertions at a and b. The individual contributions from the points z_i and infinity can be evaluated using (3.13), and one thus arrives at the system of partial differential equations

$$D_k \langle n| J^{\nu_1}(z_1) \cdots J^{\nu_m}(z_m)|\sigma\sigma\rangle = 0 \qquad \text{for } k = 1, \ldots, m\,, \tag{3.15}$$

with

$$D_k = \frac{\nu_k}{2}\left(\frac{2z_k - a - b}{\sqrt{(z_k{-}a)(z_k{-}b)}} + \sqrt{(z_k{-}a)(z_k{-}b)}\frac{\partial}{\partial z_k}\right) - n + \sum_{i\neq k} \nu_i \frac{\sqrt{(z_i{-}a)(z_i{-}b)}}{z_i - z_k}\,. \tag{3.16}$$

To solve the equations (3.15) it is convenient to pass from the complex plane with twist-field insertions at a and b, which generate a branch cut on the interval $[a, b]$, to a double cover (*cf.* [17, 45]). The letters z, w will always refer to points on the complex plane which forms the base of the double cover and ζ, ξ to points on the double cover. Since we have a single cut, the double cover has again the topology of a Riemann sphere and we fix the projection from the double cover to the base by

$$z = \frac{b-a}{4}\left(\zeta + \zeta^{-1}\right) + \frac{a+b}{2}\,. \tag{3.17}$$

Conversely, the two pre-images of a point $z \notin [a, b]$ are given by $\zeta(z)$ and $\zeta(z)^{-1}$ with

$$\zeta(z) = \frac{\left(\sqrt{z{-}a} + \sqrt{z{-}b}\right)^2}{b-a}\,. \tag{3.18}$$

For all square roots we choose the convention that there is a branch cut from $-\infty$ to 0 and that $\sqrt{1} = 1$. Then $z \neq [a, b]$ implies $|\zeta(z)| > 1$. In fact, if we write $z = x + iy + \frac{a+b}{2}$, the curve $|\zeta(z)| = r > 1$ is the following ellipse with centre $\frac{a+b}{2}$,

$$\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} = 1\,, \qquad r_x = \frac{b{-}a}{2}\frac{r^2{+}1}{2r}\,, \qquad r_y = \frac{b{-}a}{2}\frac{r^2{-}1}{2r}\,. \tag{3.19}$$

This ellipse can also be described by $|z{-}a| + |z{-}b| = \frac{b-a}{2}\left(r + r^{-1}\right)$, which shows that a and b are the two focal points of the ellipse (3.19). The significance of this parametrisation is that the contour integrals defining (1.1) will be suppressed by factors r^{-2n} when carried out along the ellipse (see section 5.3).

On the double cover, each J^--field corresponds to the pair of vertex operators $V_-(\zeta)V_+(\zeta^{-1})$, where $V_\pm(\zeta)$ denotes the vertex operator $V_{\pm 1/\sqrt{2}}(\zeta)$ of J^3-charge $\pm\frac{1}{\sqrt{2}}$. Thus one expects that

$$\begin{aligned} &\frac{\langle n|J^-(z_1)\cdots J^-(z_m)|\sigma\sigma\rangle}{\langle n|\sigma\sigma\rangle} = \\ &\quad = \left(\frac{4}{b-a}\right)^m \prod_{i=1}^m \frac{1}{\zeta_i - \zeta_i^{-1}} \left\langle \frac{n}{\sqrt{2}} \middle| V_-(\zeta_1)V_+(\zeta_1^{-1}) \cdots V_-(\zeta_m)V_+(\zeta_m^{-1}) \middle| \frac{n}{\sqrt{2}} \right\rangle, \end{aligned} \tag{3.20}$$

where $\zeta_i = \zeta(z_i)$. The right hand side can be computed in terms of the Coulomb gas expression for free boson vertex operators, and one finds

$$\frac{\langle n|J^-(z_1)\cdots J^-(z_m)\sigma^+(b)\sigma^-(a)|0\rangle}{\langle n|\sigma^+(b)\sigma^-(a)|0\rangle} = \left(\frac{4}{b-a}\right)^m \prod_{i=1}^{m} \frac{\zeta_i^{-2n}}{(\zeta_i - \zeta_i^{-1})^2} \prod_{i>j} \left(\frac{\zeta_i - \zeta_j}{\zeta_i\zeta_j - 1}\right)^2 . \quad (3.21)$$

It is straightforward to check that this indeed solves (3.15); to this end it is useful to rewrite the differential operator (3.16) also in terms of the ζ-variables, using $\zeta - \zeta^{-1} = \frac{4}{b-a}\sqrt{(z-a)(z-b)}$ and $\frac{\partial}{\partial\zeta} = \frac{b-a}{4}(1-\zeta^{-2})\frac{\partial}{\partial z}$. Of course, (3.15) only determines (3.21) up to a constant. This constant can be found recursively using (3.2),

$$\langle n{+}1|J^-(z_1)\cdots J^-(z_m)|\sigma\sigma\rangle = \int_{\mathcal{C}_\infty} w^{2n+1}\langle n|J^-(w)J^-(z_1)\cdots J^-(z_m)|\sigma\sigma\rangle \frac{dw}{2\pi i}\,. \qquad (3.22)$$

This determines the overall constant for a correlator with $m{+}1$ insertions of J^- in terms of a correlator with only m insertions. Finally, the expressions with no insertions of J^- has already been given in (3.7). This procedure is the origin of the factor $\left(\frac{4}{b-a}\right)^m$ in (3.20) and (3.21).

A correlator where some of the $J^-(z_i)$ insertions have been replaced by $J^+(z_i)$ insertions can be obtained by continuing z_i through the branch cut $[a,b]$, which amounts to replacing $\zeta_i \to \zeta_i^{-1}$ in (3.21).

4. Properties of the operator $I_\lambda(a,b)$

In the previous section we have collected some information about the structure of the twist field correlators. Now we want to study the correlation functions of $I_\lambda(a,b)$ defined in (1.1).

4.1 Monodromy and operator product expansion of su(2)-currents

Before computing the monodromy of the su(2)-currents $J^\pm$, J^3 in the presence of $I_\lambda(a,b)$ let us consider a slightly more general situation. Define the operator

$$B(a,b) = \exp\left(\int_a^b F(x)\,dx\right), \qquad (4.1)$$

where $F(x)$ is a linear combination of holomorphic functions multiplying chiral fields (and we assume that the operator product expansion of $F(x_1)$ and $F(x_2)$ does not have any poles so that no normal ordering prescription is necessary). Consider the analytic continuation of a chiral field $\phi(z)$ around the point b (see figure 3).

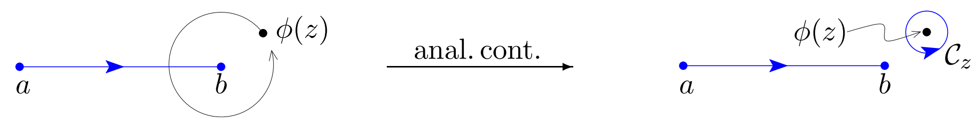

Figure 3: Analytic continuation of a chiral field around the point b.

It is then clear that under the analytic continuation of figure 3, the monodromy of $\phi(z)$ is

$$\phi(z) \longrightarrow \exp\left(2\pi i \int_{\mathcal{C}_z} F(x) \frac{dx}{2\pi i}\right) \phi(z)\,. \tag{4.2}$$

In the case of $I_\lambda(a,b)$ we have $F(x) = \frac{\lambda}{2\pi} J^+(x)$. Acting on the state $|\phi\rangle$ corresponding to the field $\phi(z)$, the monodromy (4.2) then reads $|\phi\rangle \to \exp(i\lambda J_0^+)|\phi\rangle$. Representing the linear combination $|\phi\rangle = \alpha|J^+\rangle + \beta|J^3\rangle + \gamma|J^-\rangle$ by the vector (α,β,γ), the monodromy of the su(2)-currents around the endpoint b of $I_\lambda(a,b)$ is given by the matrix

$$M_\lambda(b) = \begin{pmatrix} 1 & -i\lambda & \lambda^2 \\ 0 & 1 & 2i\lambda \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.3}$$

For example, $\exp(i\lambda J_0^+)|J^3\rangle = (J^3_{-1} + i\lambda J_0^+ J^3_{-1})|0\rangle = |J^3\rangle - i\lambda|J^+\rangle$. Since the currents are single valued on $\mathbb{C} - [a,b]$, the monodromy around a is inverse to that around b so that $M_\lambda(a) = M_\lambda(b)^{-1} = M_{-\lambda}(b)$.

The above calculation can be repeated for the stress tensor $T(z)$ and one finds that due to $[J_0^+, L_m] = 0$, the stress tensor is single valued across $[a,b]$. Translating this observation back into matrix model language (as briefly described in section 1.1), gives a way to derive the quadratic loop equation of the matrix model from the free boson conformal field theory [29, 42, 33].

To analyse the singularities of the su(2)-currents close to the points a and b it is helpful to introduce the following combinations of fields

$$\begin{aligned} \widehat{J}^+(z) &= J^+(z) \\ \widehat{J}^3(z) &= J^3(z) + \frac{\lambda}{2\pi}\ln\frac{z-b}{z-a}\, J^+(z) \\ \widehat{J}^-(z) &= J^-(z) - \frac{2\lambda}{2\pi}\ln\frac{z-b}{z-a}\, J^3(z) - \left(\frac{\lambda}{2\pi}\ln\frac{z-b}{z-a}\right)^2 J^+(z)\,. \end{aligned} \tag{4.4}$$

Using (4.3) one verifies that $\widehat{J}^\pm(z)$ and $\widehat{J}^3(z)$ are single valued in the presence of a single insertion of $I_\lambda(a,b)$. In particular, they can be expanded in integer modes

$$\widehat{J}^c(z) = \sum_{m\in\mathbb{Z}} (z-p)^{-m-1} \widehat{J}^c_{m;p}\,, \qquad \text{where} \qquad \widehat{J}^c_{m;p} = \int (z-p)^m \widehat{J}^c(z) \frac{dz}{2\pi i}\,. \tag{4.5}$$

Here $c \in \{+,3,-\}$ and p equals a or b. We would like to establish the following statement:

(S) Suppose $\lim_{z\to b}(z-b)J^-(z)I_\lambda(a,b)$ is finite inside any correlator, and there exists some $N > 0$ such that $\lim_{z\to b}(z-b)^N J^+(z)I_\lambda(a,b)$ and $\lim_{z\to b}(z-b)^N J^3(z)I_\lambda(a,b)$ are zero. Then

$$\widehat{J}^\pm_{m;b} I_\lambda(a,b) = 0 = \widehat{J}^3_{m;b} I_\lambda(a,b) \qquad \text{for } m > 0\,, \qquad \widehat{J}^+_{0;b} I_\lambda(a,b) = 0 = \widehat{J}^3_{0;b} I_\lambda(a,b)\,. \tag{4.6}$$

Together with the analogous statement for a, this means that the fields at the end points a and b are in fact highest weight with respect to the $\widehat{J}^c$ action.

The vanishing of the J^+-zero mode implies in particular that $J^+(z)I_\lambda(a,b)$ is regular for $z \to b$. Of course, this also follows immediately when writing out the integrals in (1.1) and using that the operator product expansion of J^+ with itself is regular. However, in section 5.5 we will need to apply statement (S) to $S_\lambda(a,b)$ instead of $I_\lambda(a,b)$, so we will present the argument in a form which will be valid also then.

To establish (S), start by expressing J^- in terms of the single valued combinations (4.4),

$$J^-(z) = \widehat{J}^-(z) + \frac{2\lambda}{2\pi}\ln\frac{z-b}{z-a}\widehat{J}^3(z) - \left(\frac{\lambda}{2\pi}\ln\frac{z-b}{z-a}\right)^2\widehat{J}^+(z)\,. \tag{4.7}$$

Denote by $C(\phi)$ a correlator involving ϕ and $I_\lambda(a,b)$, as well as any number of other fields. Expanding the right hand side of (4.7) in terms of modes around b then gives

$$C(J^-(z)) = \sum_{m=-\infty}^{N}(z-b)^{-m-1}\left(C(\widehat{J}^-_{m;b}) + \frac{2\lambda}{2\pi}\ln\frac{z-b}{z-a}C(\widehat{J}^3_{m;b}) - \left(\frac{\lambda}{2\pi}\ln\frac{z-b}{z-a}\right)^2 C(\widehat{J}^+_{m;b})\right). \tag{4.8}$$

The summation is truncated by the assumption on the limit $z \to b$ of J^+ and J^3. Evaluating the conditions $\lim_{z\to b}(z-b)^{m+1}J^-(z)I_\lambda(a,b) = 0$ for $m = N, N-1, \ldots, 1$ gives $C(\widehat{J}^{\pm,3}_{m;b}) = 0$ for that range of m. Finally, for $m = 0$ we get $C(\widehat{J}^+_{0;b}) = 0 = C(\widehat{J}^3_{0;b})$. Since $C(\ldots)$ was an arbitrary correlator, this implies (4.6).

In order to establish the highest weight relations (4.6) for $I_\lambda(a,b)$ we still have to verify that the conditions of the statement (S) are met. For J^+ this is obvious, but for J^3 and J^- this requires a short calculation which is given in appendix A.1.

Finally, let us show that the endpoints of $I_\lambda(a,b)$ obey the Virasoro highest weight condition for weight zero. To this end, instead of expressing the stress tensor $T(z)$ as in (2.3) we use the single valued fields (4.4). Computing the operator product expansion of $\widehat{J}^3$ with itself to order $O(z-w)$ one finds

$$T(w) = \lim_{z\to w}\left(\widehat{J}^3(z)\widehat{J}^3(w) - \frac{1/2}{(z-w)^2}\right) + \frac{\lambda}{2\pi}\frac{b-a}{(w-a)(w-b)}\widehat{J}^+(w)\,, \tag{4.9}$$

which in terms of modes around b reads

$$L_{m;b} = \sum_{k\in\mathbb{Z}} :\widehat{J}^3_{k;b}\widehat{J}^3_{m-k;b}: + \frac{\lambda}{2\pi}\sum_{k=0}^{\infty}(a-b)^{-k}\widehat{J}^+_{m+k;b}\,. \tag{4.10}$$

Together with (4.6) this immediately implies that $L_{m;b}I_\lambda(a,b) = 0$ for $m \geq 0$. For a instead of b one finds the same result.

4.2 Zero-point function

Up to now we have kept the parameter λ arbitrary. But just as was the case for twist fields, we can restrict our attention to the case $\lambda = 1$. The results for general values of λ are then obtained via the identity

$$I_\lambda(a,b)|0\rangle = e^{\ln(\lambda)J^3_0}I_1(a,b)|0\rangle\,. \tag{4.11}$$

To see this write $I_\lambda(a,b)$ and $I_1(a,b)$ as a sum over J^+ integrations and use $[J^3_0, J^+(z)] = J^+(z)$ which leads to $\exp(\ln(\lambda)J^3_0)J^+(z) = \lambda J^+(z)\exp(\ln(\lambda)J^3_0)$. Moving the exponential in (4.11) past the J^+ insertions in the expansion of $I_1(a,b)$ gives a factor of λ for each insertion.

The correlator we would like to compute is $\langle n|I_1(a,b)|0\rangle$. From the previous section we know that the endpoints of $I_1(a,b)$ behave as Virasoro primary fields of weight zero. Applying a rescaling and a translation yields

$$\langle n|I_1(a,b)|0\rangle = (b-a)^{n^2} 2^{-n^2} \langle n|I_1(-1,1)|0\rangle\,. \tag{4.12}$$

The correlator on the right hand side can be computed by explicit integration using orthogonal polynomials on the interval $[-1,1]$, i.e. the Legendre polynomials (see e.g. [11, 22] for an introduction to the method of orthogonal polynomials), or by directly using Selberg's integral (see e.g. [37, chapter 17]). The result is

$$\begin{aligned}\langle n|I_1(-1,1)|0\rangle &= \frac{1}{(2\pi)^n\, n!}\int_{-1}^{1} dx_1\cdots dx_n\ \langle n|J^+(x_1)\cdots J^+(x_n)|0\rangle \\ &= \frac{1}{(2\pi)^n\, n!}\int_{-1}^{1} dx_1\cdots dx_n\,\Delta(x)^2 = (2\pi)^{-n}\,2^{n^2}\det(H_n)\,.\end{aligned} \tag{4.13}$$

In the first step, the definition (1.1) has been substituted, in the second step the Coulomb gas expression for the integrand has been written in terms of the Vandermonde determinant $\Delta(x) = \prod_{i>j}(x_i - x_j)$. In the result, H_n is the $n{\times}n$-Hilbert matrix $(H_n)_{ij} = (i+j-1)^{-1}$. Its determinant is given by

$$\det(H_n) = 2^{-n^2}\prod_{k=0}^{n-1}\frac{2}{2k+1}\left(\frac{2^k\,k!^2}{(2k)!}\right)^2 = \frac{2^{n-2n^2}\pi^{n+1/2}}{\Gamma(n+\frac{1}{2})}\left(\frac{G(n+1)\,G(\frac{1}{2})}{G(n+\frac{1}{2})}\right)^2. \tag{4.14}$$

Here $G(z)$ is the Barnes function, which is defined by $G(z+1) = \Gamma(z)G(z)$ and $G(1) = 1$ together with a convexity condition. The behaviour of $\det(H_n)$ for large-n can now be obtained from the large z expansion of $G(z)$. The latter can be found in [1, eq. (28)] (in the preprint (v1), the minus in front of the sum should be a plus) or in [35, eq. (2.38)]. In this way, we finally obtain

$$\langle n|I_1(a,b)|0\rangle = (b-a)^{n^2}4^{-n^2}n^{-\frac{1}{4}}2^{\frac{1}{12}}\,e^{3\zeta'(-1)}\exp\Big(-\frac{1}{64}n^{-2} + \frac{1}{256}n^{-4} + O(n^{-6})\Big). \tag{4.15}$$

Comparing to the correlator of two twist fields (3.7) we observe that $\sigma_+(b)\sigma_-(a)$ does correctly reproduce the leading term in (4.15) in the sense that

$$\lim_{n\to\infty} n^{-2}\ln\langle n|\sigma_+(b)\sigma_-(a)|0\rangle = \lim_{n\to\infty} n^{-2}\ln\langle n|I_1(a,b)|0\rangle\,. \tag{4.16}$$

4.3 One-point function

After computing the zero-point function we will turn to the correlators $\langle n|J^c(z)I_1(a,b)|0\rangle$ for $c \in \{+,3,-\}$. The result can again be obtained by matrix model techniques using orthogonal polynomials (see e.g. [37, chapter 22] or [22, section 4]), which amount to explicitly computing the relevant integrals. Another method more in the spirit of conformal field theory is to use that the $h = 1$, $c = 1$ Virasoro highest weight representation has a null vector at level three. Denoting the highest weight vector in this representation by $|J\rangle$ the null vector $|\eta\rangle$ is

$$|\eta\rangle = \Big(3L_{-3} - 2L_{-1}L_{-2} + \frac{1}{2}L_{-1}L_{-1}L_{-1}\Big)\Big|J\Big\rangle = 0\,. \tag{4.17}$$

Since the three su(2) currents $J^c(z)$ are Virasoro primaries of weight one, the three correlators $\langle n|J^c(z)I_1(a,b)|0\rangle$ will all solve the same third order differential equation in z, obtained from the null vector $|\eta\rangle$. To compute this differential equation, recall from section 3.1 that from the point of view of the Virasoro algebra there is no difference between an insertion of $I_\lambda(a,b)$ and a product $\phi(b)\phi(a)$ of Virasoro primary fields ϕ with conformal weight zero. Specialising to $a = -1$ and $b = 1$, the differential equation is then found to be (see [12, section 8.3] for more details on null-vector computations)

$$\begin{aligned} 0 &= \langle n|\eta(z)\phi(1)\phi(-1)|0\rangle \\ &= \Big\{\frac{1}{2}\partial_z^3 + \frac{4z}{z^2-1}\partial_z^2 + \frac{5z^2-1}{(z^2-1)^2}\partial_z - \frac{2(n^2-1)}{z^2-1}\Big(\frac{z}{z^2-1} + \partial_z\Big)\Big\}\langle n|J(z)\phi(1)\phi(-1)|0\rangle\,. \end{aligned} \tag{4.18}$$

The space of solutions to this equation is three-dimensional. The elements in this space describing the three functions $\langle n|J^c(z)I_1(-1,1)|0\rangle$ have to be identified from their behaviour at the singular points -1, 1, ∞.

In any case, using either orthogonal polynomials or Virasoro null vectors, the final result for the one-point functions is

$$\begin{aligned} \langle n|J^+(z)I_1(-1,1)|0\rangle &= \pi n\Big(P_{n-1}(z)P_n'(z) - P_n(z)P_{n-1}'(z)\Big)\langle n|I_1(-1,1)|0\rangle \\ \langle n|J^3(z)I_1(-1,1)|0\rangle &= n\Big(P_{n-1}(z)Q_n'(z) - P_n(z)Q_{n-1}'(z)\Big)\langle n|I_1(-1,1)|0\rangle \\ \langle n|J^-(z)I_1(-1,1)|0\rangle &= -\frac{n}{\pi}\Big(Q_{n-1}(z)Q_n'(z) - Q_n(z)Q_{n-1}'(z)\Big)\langle n|I_1(-1,1)|0\rangle\,. \end{aligned} \tag{4.19}$$

In these equations, $P_n(z)$ is the n'th Legendre polynomial and $Q_n(z)$ the n'th Legendre function of the second kind. The first few are (chosen such that they are real for $z \in \mathbb{R} - [-1,1]$)

$$Q_0(z) = \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right), \qquad Q_1(z) = \frac{z}{2}\ln\left(\frac{z+1}{z-1}\right) - 1\,, \qquad Q_2(z) = \frac{3z^2-1}{4}\ln\left(\frac{z+1}{z-1}\right) - \frac{3z}{2}\,. \tag{4.20}$$

One can verify that the functions (4.19) solve (4.18) and their monodromy around the point 1 is given by (4.3).

To understand the large-n behaviour of the one-point functions, it is convenient to write $P_n(z)$ and $Q_n(z)$ for $z \notin [-1,1]$ in terms of hypergeometric functions as [27, eq. (8.723)]

$$P_n(z) = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)}\frac{\zeta^{n+1}}{\sqrt{\zeta^2-1}}{}_2F_1\Big(\frac{1}{2},\frac{1}{2};\frac{1}{2}-n;\frac{-1}{\zeta^2-1}\Big)$$

$$Q_n(z) = \frac{\sqrt{\pi}\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} \frac{\zeta^{-n}}{\sqrt{\zeta^2-1}} {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}+n; \frac{-1}{\zeta^2-1}\Big). \tag{4.21}$$

Here, $\zeta = \zeta(z)$ is given by (3.18) with $a = -1$, $b = 1$. The large-n expansion of the correlators (4.19) is then found by writing out the definition of the hypergeometric functions (4.21) as a power series and expanding the Gamma-functions,

$$\begin{aligned}
\frac{\langle n|J^+(z)I_1(-1,1)|0\rangle}{\langle n|I_1(-1,1)|0\rangle} &= \frac{2\,\zeta^{2n}}{(\zeta-\zeta^{-1})^2}\Big(1 + \frac{1}{4}\frac{\zeta+\zeta^{-1}}{\zeta-\zeta^{-1}}n^{-1} + \\
&\qquad + \frac{1}{32}\frac{(1+5\zeta^2)(1+5\zeta^{-2})}{(\zeta-\zeta^{-1})^2}n^{-2} + O(n^{-3})\Big) \\
\frac{\langle n|J^3(z)I_1(-1,1)|0\rangle}{\langle n|I_1(-1,1)|0\rangle} &= \frac{2n}{\zeta-\zeta^{-1}}\Big(1 - \frac{1}{2(\zeta-\zeta^{-1})^2}n^{-2} + O\left(n^{-3}\right)\Big) \\
\frac{\langle n|J^-(z)I_1(-1,1)|0\rangle}{\langle n|I_1(-1,1)|0\rangle} &= \frac{2\,\zeta^{-2n}}{(\zeta-\zeta^{-1})^2}\Big(1 - \frac{1}{4}\frac{\zeta+\zeta^{-1}}{\zeta-\zeta^{-1}}n^{-1} + \\
&\qquad + \frac{1}{32}\frac{(1+5\zeta^2)(1+5\zeta^{-2})}{(\zeta-\zeta^{-1})^2}n^{-2} + O\left(n^{-3}\right)\Big).
\end{aligned} \tag{4.22}$$

Comparing to (3.9) and (3.21) (specialised to $a = -1$ and $b = 1$) we see again that the leading behaviour in $1/n$ is the same for $I_1(a, b)$ and the twist fields $\sigma_+(b)\sigma_-(a)$.

In fact not only the leading term in the expansions (4.22) has the $\mathbb{Z}_2$-symmetry $J^\pm \to J^\mp$ and $J^3 \to -J^3$ upon analytically continuing z through the branch cut $[-1, 1]$, but this monodromy is retained at any finite order in the expansions (4.22). It is only after summing all terms that the 'correct' monodromy (4.3) is recovered. For example, at finite values of n, the correlator $\langle n|J^+(z)I_1(-1,1)|0\rangle$ is a single valued function of z (in fact, a polynomial), while in the $1/n$-expansion it has a branch cut on $[-1, 1]$. Recall from (1.7) that this conformal field theory correlator is related to the correlator $Z^{\text{well}}_{\text{mm}}[\det(z-M)^2]^{(n)}$ in the hermitian one-matrix model, and hence the latter shares with the correlator of $J^+(z)$ the property that the monodromy of the individual terms in the $1/n$-expansion differs from the monodromy of the complete expression. This is a manifestation of Stokes' phenomenon as mentioned at the end of section 1.1.

5. Writing $I_\lambda(a, b)$ in terms of twist fields

In sections 3 and 4 we have collected some properties of su(2)-twist fields and of the operator $I_\lambda(a, b)$. We have seen that in sectors of large J^3-charge, $I_\lambda(a, b)$ behaves very similar to a product $\sigma_+(b)\sigma_-(a)$ of twist fields. Motivated by these observations one can now try to find an alternative expression for the operator $I_\lambda(a, b)$ in terms of appropriately dressed twist fields.

To achieve this, we make an ansatz $S_\lambda(a, b)$ involving twist fields and J^--integrals in section 5.1, and show that it can reproduce the correct monodromy. That the su(2)-currents have the same monodromy for $I_\lambda(a, b)$ and $S_\lambda(a, b)$ is the first piece of evidence for the proposed identity $I_\lambda(a, b) = S_\lambda(a, b)$. The J^- integrals in $S_\lambda(a, b)$ are divergent and need to be regulated; this is done in section 5.2. Next, in section 5.3 we investigate

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the large-n behaviour of $S_\lambda(a,b)$ and find that it is given in terms of a product of twist fields $\sigma_+(b)\sigma_-(a)$. This is the second piece of evidence for $I_\lambda(a,b) = S_\lambda(a,b)$, since we saw, in sections 4.2 and 4.3, that the large-n limit of the zero- and one-point function in the presence of $I_\lambda(a,b)$ agrees with the corresponding twist field correlators. The requirement that $S_\lambda(a,b)$ should have the correct monodromy still left an (a,b)-dependent normalisation factor undetermined, which is fixed in section 5.4 by matching it against the leading large-n behaviour of $\langle n|I_\lambda(a,b)|0\rangle$. Finally, in section 5.5 we present the third supporting evidence by checking, to the extend that we were able to calculate it, that the su(2)-currents have the same singularities in the presence of $S_\lambda(a,b)$ as were seen for $I_\lambda(a,b)$ in section 4.1.

5.1 Reproducing the monodromy of $I_\lambda(a,b)$

Our first task will be to modify the product $\sigma_+(b)\sigma_-(a)$ in such a way that instead of the $\mathbb{Z}_2$-monodromy, we find the monodromy (4.3) for the su(2)-currents. As a motivation for the ansatz below, compare the definition (1.1) of $I_\lambda(a,b)$ to the expression (2.20) for the product $\sigma_+(b)\sigma_-(a)$. It seems one can go from the latter to the former by 'subtracting' the J^--contribution to the integrals. In this spirit, define an operator

$$\tilde{S}(a,b) = C(a,b)\left[\sigma_{+\gamma}(b)\exp\left(\frac{\alpha}{2\pi}\int_{\mathcal{C}_1} J^-(x)dx\right)\exp\left(\frac{\beta}{2\pi}\int_{\mathcal{C}_2} J^-(x)dx\right)\sigma_{-\gamma}(a)\right]_{\rm reg}. \quad (5.1)$$

The contours $\mathcal{C}_{1,2}$ are as shown in figure 1, $C(a,b)$ is a $\mathbb{C}$-valued function and α, β, γ are constants. Let us postpone the discussion of the regulator to section 5.2; in any case it will be defined in such a way that it does not affect the monodromy.

The monodromy of the su(2)-currents around the point b of $\tilde{S}(a,b)$ is a product of three terms, two from the exponentiated J^--integrals and one from the $\mathbb{Z}_2$-monodromy of the twist fields. In terms of the basis used in (4.3) the monodromy is

$$\begin{aligned}\tilde{M}(b) &= \begin{pmatrix} 1 & 0 & 0 \\ -2i\beta & 1 & 0 \\ \beta^2 & i\beta & 1\end{pmatrix}\begin{pmatrix} 0 & 0 & \gamma^2 \\ 0 & -1 & 0 \\ \gamma^{-2} & 0 & 0\end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -2i\alpha & 1 & 0 \\ \alpha^2 & i\alpha & 1\end{pmatrix} \\ &= \begin{pmatrix} \alpha^2\gamma^2 & i\alpha\gamma^2 & \gamma^2 \\ 2i\alpha(1-\alpha\beta\gamma^2) & 2\alpha\beta\gamma^2-1 & -2i\beta\gamma^2 \\ \gamma^{-2}(1-\alpha\beta\gamma^2)^2 & i\beta(\alpha\beta\gamma^2-1) & \beta^2\gamma^2\end{pmatrix}. \end{aligned} \quad (5.2)$$

We see that this matches with the monodromy $M_\lambda(b)$ of $I_\lambda(a,b)$ in (4.3) for precisely two choices of the three parameters α, β, γ, namely $\alpha = \beta = -\lambda^{-1}$ and $\gamma = \pm\lambda$. We will choose $\gamma = \lambda$.

5.2 Regulating the J^--integrals

To regulate the J^- integrals in (5.1) we will first introduce a cutoff ε and then present a subtraction scheme which we conjecture to give a finite $\varepsilon \to 0$ limit. In fact, we will regulate (5.1) for the choice $\gamma = 1$, which will give the operator $S_\lambda(a,b)$ in (1.11) for $\lambda = 1$. General values of λ will then be obtained as in (4.11).

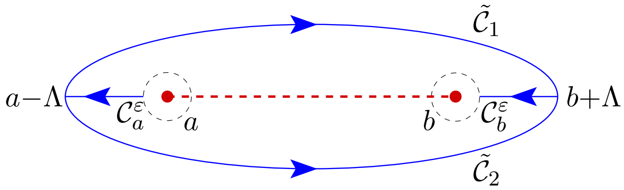

Figure 4: Contour for the regularised J^- integrals.

The ε-cutoff is imposed simply by changing the integration contours $\mathcal{C}_{1,2}$ to approach the points a, b only up to a distance ε. More precisely, fix a small positive constant Λ as well as a value $\varepsilon \ll \Lambda$. Consider the integration contours $\tilde{\mathcal{C}}_{1,2}$ and $\mathcal{C}^\varepsilon_{a,b}$ defined as in figure 4. The dashed circles around the points a and b have radius ε so that $\mathcal{C}^\varepsilon_a = [a{-}\Lambda, a{-}\varepsilon]$ and $\mathcal{C}^\varepsilon_b = [b{+}\varepsilon, b{+}\Lambda]$, with orientations as indicated. Indeed, for $\varepsilon = 0$ this is just a deformation of the contours $\mathcal{C}_{1,2}$ defined in figure 1. Instead of integrating $J^-(x)$ define a field $\rho_t(x)$ as

$$\rho_t(x) = tJ^-(x) - f(t)\langle J^-(x)\rangle \mathbf{1}\,, \qquad \text{where} \qquad \langle J^-(x)\rangle \equiv \frac{\langle 0|J^-(x)\sigma_+(b)\sigma_-(a)|0\rangle}{\langle 0|\sigma_+(b)\sigma_-(a)|0\rangle}\,. \tag{5.3}$$

Here t is a complex parameter and $f(t) = t + f_2t^2 + f_3t^3 + \dots$ is a power series in t. On the segments $\mathcal{C}^\varepsilon_a$, $\mathcal{C}^\varepsilon_b$ and $\tilde{\mathcal{C}}_{1,2}$ define the following integrated fields,

$$\begin{aligned} U^{\varepsilon,\Lambda}_{+,t}(b) &= \exp\left(-\frac{1}{\pi}\int_{\mathcal{C}^\varepsilon_b}\rho_t(x)dx\right)\sigma_+(b) \\ U^{\varepsilon,\Lambda}_{-,t}(a) &= \exp\left(-\frac{1}{\pi}\int_{\mathcal{C}^\varepsilon_a}\rho_t(x)dx\right)\sigma_-(a) \\ V^\Lambda_t(a,b) &= C(a,b)\exp\left(-\frac{1}{2\pi}\int_{\tilde{\mathcal{C}}_1}\rho_t(x)dx\right)\exp\left(-\frac{1}{2\pi}\int_{\tilde{\mathcal{C}}_2}\rho_t(x)dx\right). \end{aligned} \tag{5.4}$$

In $U^{\varepsilon,\Lambda}_{\pm,t}$ there is a factor of $\frac{1}{\pi}$ instead of $\frac{1}{2\pi}$ because in deforming the contour of figure 1 to that of figure 4, the segments $C^\varepsilon_{a,b}$ are traversed twice.

The subtraction scheme now consists of expanding the operators $U^{\varepsilon,\Lambda}_{\pm,t}$ as a formal power series in t and demanding that each term in the expansion has a finite limit as $\varepsilon \to 0$. This procedure will result in conditions determining the constants f_2, f_3, $\dots$. It is not at all obvious that such a solution exists, and we have no proof that it can be done to all orders in t. In appendix A.2 we verify that the subtraction scheme (5.4) works at least to order t^3, with $f(t) = t - \frac{1}{\pi}t^2 + \frac{1}{6}t^3 + O(t^4)$. To proceed we will assume:

(A1) There exists a function $f(t)$ such that each order in the expansion of the operators $U^{\varepsilon,\Lambda}_{+,t}(b)$ and $U^{\varepsilon,\Lambda}_{-,t}(a)$ in powers of t has a finite limit as $\varepsilon \to 0$.

Using (A1), we can define the operator $S_1(a,b)$ in terms of $U^{\varepsilon,\Lambda}_{\pm,t}$ as

$$S_1(a,b) = V^\Lambda_1(a,b)S^{\text{pert}}_{+,\Lambda}(b)S^{\text{pert}}_{-,\Lambda}(a)\,, \tag{5.5}$$

where

$$S^{\text{pert}}_{+,\Lambda}(b) = \left(\lim_{\varepsilon\to 0}U^{\varepsilon,\Lambda}_{+,t}(b)\right)_{t=1}, \qquad S^{\text{pert}}_{-,\Lambda}(a) = \left(\lim_{\varepsilon\to 0}U^{\varepsilon,\Lambda}_{-,t}(a)\right)_{t=1}. \tag{5.6}$$

Finally we then obtain

$$S_\lambda(a,b)|0\rangle = e^{\ln(\lambda)J_0^3} S_1(a,b)|0\rangle\,. \tag{5.7}$$

A number of comments are in order. First, the abbreviation 'pert' in (5.6) stands for 'perturbative', a qualifier that will be justified in the next section. Second, while the decomposition (5.5) of $S_1(a,b)$ will be useful in the following, we can equivalently write it in a form that more closely resembles (5.1),

$$S_1(a,b) = C(a,b)\Big(\lim_{\varepsilon\to 0}\sigma_+(b)\exp\Big(-\frac{1}{2\pi}\int_{\mathcal{C}_1^\varepsilon}\rho_t(x)dx\Big)\exp\Big(-\frac{1}{2\pi}\int_{\mathcal{C}_2^\varepsilon}\rho_t(x)dx\Big)\sigma_-(a)\Big)_{t=1}\,. \tag{5.8}$$

Here, $\mathcal{C}_1^\varepsilon$ is the contour obtained by joining $\mathcal{C}_a^\varepsilon$, $\tilde{\mathcal{C}}_1$ and $\mathcal{C}_b^\varepsilon$, and $\mathcal{C}_2^\varepsilon$ is obtained by joining $\mathcal{C}_a^\varepsilon$, $\tilde{\mathcal{C}}_2$ and $\mathcal{C}_b^\varepsilon$. In the form (5.8) it is also apparent that $S_1(a,b)$ has the monodromy (5.2), since $\rho_1(x)$ differs from $J^-(x)$ only by a central term.

Third, it is clear from the definition that $S^{\rm pert}_{+,\Lambda}(b)$ and $S^{\rm pert}_{-,\Lambda}(a)$ are coherent states in the $\mathbb{Z}_2$ twisted representations generated by σ_+ and σ_-, respectively. Coherent states in twisted representations also appeared in relation to matrix models in [33, 13].

5.3 Behaviour of $S_\lambda(a,b)$ at large n

In section 4.3 we have seen that the monodromy of $I_\lambda(a,b)$ around a or b is entirely due to 'non-perturbative' effects, i.e. that to any finite order in $1/n$, the monodromy is just given by the $\mathbb{Z}_2$ twist (1.9). We now want to show that the behaviour of $S_\lambda(a,b)$ is the same; more explicitly, we shall show that (see (1.14))

$$\langle n|(\text{fields})S_\lambda(a,b)|0\rangle = \langle n|(\text{fields})S_\lambda^{\rm trunc}(a,b)|0\rangle\,\big(1+O(r^{-2n})\big)\,, \tag{5.9}$$

for some $r>0$, where $S_\lambda^{\rm trunc}(a,b)$ has the same monodromy as the two $\mathbb{Z}_2$-twist fields. We will again only treat the case $\lambda=1$; for the general case the reasoning is analogous due to (5.7).

To define $S_1^{\rm trunc}(a,b)$ we choose the contours $\tilde{\mathcal{C}}_{1,2}$ in figure 4 to lie on the ellipse (3.19) of constant $|\zeta|$ passing through $a-\Lambda$ and $b+\Lambda$. This amounts to choosing $r=|\zeta|$ to be

$$r = 1+\frac{2\Lambda}{b{-}a}\left(1+\sqrt{1+\frac{b{-}a}{\Lambda}}\right). \tag{5.10}$$

Then define the function

$$D_t^\Lambda(a,b) = C(a,b)\exp\left(\frac{1}{\pi}f(t)\int_{\tilde{\mathcal{C}}_1}\langle J^-(x)\rangle dx\right)\,, \tag{5.11}$$

and set

$$S_1^{\rm trunc}(a,b) = D_1^\Lambda(a,b)S^{\rm pert}_{+,\Lambda}(b)\,S^{\rm pert}_{-,\Lambda}(a)\,. \tag{5.12}$$

The claim (5.9) is then implied by the following statement: let $(F)=\prod_{i=1}^{|F|}J^{c_i}(w_i)$ abbreviate a product of su(2)-currents. Then we have

$$\langle n|(F)\big(V_t^\Lambda(a,b)-D_t^\Lambda(a,b)\mathbf{1}\big)\eta_+(b)\eta_-(a)|0\rangle = \langle n|(F)\eta_+(b)\eta_-(a)|0\rangle\cdot O(r^{-2n})\,, \tag{5.13}$$

where $\eta_\pm(z)$ are fields corresponding to states in the $\mathbb{Z}_2$-twisted representations. This is sufficient to establish (5.9) since the fields $S^{\text{pert}}_{\pm,\Lambda}(z)$ correspond to (coherent) states in the $\mathbb{Z}_2$-twisted representations (and thus can play the roles of $\eta_\pm$).

To prove (5.13), first note that it is enough to consider products $(F) = \prod_{i=1}^{|F|} J^{\nu_i}(w_i)$ with $\nu_i = \pm$, because J^3 insertions can be obtained in the operator product expansion of J^+ and J^-. To express twist-field descendents, define the modes $M_{r,s}$ as

$$M_{r,s} = \int_{\mathcal{C}_{ab}} (z-a)^r (z-b)^s J^3(z) \frac{dz}{2\pi i}\,, \qquad r,s \in \mathbb{Z} + \frac{1}{2}\,, \tag{5.14}$$

where $\mathcal{C}_{ab}$ is a contour encircling a and b. Since we are at level $k = 1$, the entire $\mathbb{Z}_2$-twisted representation is generated by acting with modes J^3_r, $r \in \mathbb{Z}_{<0}+\frac{1}{2}$ on $\sigma_\pm$. Correspondingly, the product $\eta_+(b)\eta_-(a)$ can be written as a linear combination of appropriate products $\prod_i M_{r_i,s_i}\sigma_+(a)\sigma_-(b)$, with $r_i+s_i \le 0$. Next, note that by definition,

$$V^\Lambda_t(a,b) - D^\Lambda_t(a,b)\mathbf{1} = D^\Lambda_t(a,b) \sum_{m=1}^{\infty} \frac{1}{m!} \left(\frac{-t}{2\pi}\right)^m \int_{\tilde{\mathcal{C}}_1+\tilde{\mathcal{C}}_2} J^-(x_1)\cdots J^-(x_m)\, dx_1 \cdots dx_m\,. \tag{5.15}$$

Upon inserting (5.15) into (5.13) one obtains integrands of the form

$$I_m = \left\langle n \middle| (F) J^-(x_1)\cdots J^-(x_m) \prod_{i=1}^{M} M_{r_i,s_i} \middle| \sigma\sigma \right\rangle, \qquad |\sigma\sigma\rangle = \sigma_+(b)\sigma_-(a)|0\rangle\,. \tag{5.16}$$

Using the commutator $[M_{r,s}, J^\pm(x)] = \pm(x-a)^r(x-b)^s J^\pm(x)$, as well as $\langle n|M_{r,s} = \delta_{r+s,0}n\langle n|$ (expand $M_{r,s}$ in integer modes of J^3 and use the condition $r+s \le 0$) we can write

$$\begin{aligned} I_m &= A\langle n|(F)J^-(x_1)\cdots J^-(x_m)|\sigma\sigma\rangle \\ A &= \prod_{i=1}^{M}\Bigg(-\sum_{j=1}^{|F|} \nu_j (w_j-a)^{r_i}(w_j-b)^{s_i} + \sum_{j=1}^{m} (x_j-a)^{r_i}(x_j-b)^{s_i} + n\delta_{r_i+s_i,0}\Bigg). \end{aligned} \tag{5.17}$$

To proceed we need the following two estimates on the large-n behaviour of correlators,

$$\frac{A\langle n|(F)|\sigma\sigma\rangle}{\langle n|(F)\prod_{i=1}^{M} M_{r_i,s_i}|\sigma\sigma\rangle} = (\text{const}) + O(n^{-1})\,, \quad \frac{\langle n|(F)J^-(x_1)\cdots J^-(x_m)|\sigma\sigma\rangle}{\langle n|(F)|\sigma\sigma\rangle} = O(r^{-2nm})\,. \tag{5.18}$$

For the first estimate note that the correlator in the denominator produces a factor given by A as in (5.17) but with $m = 0$. To see the second estimate, insert the explicit solution (3.21) for the correlators. For each $J^-(x_k)$ insertion there is a factor of $\zeta(x_k)^{-2n}$ in the numerator which, since the x_k lie on the contour $\tilde{\mathcal{C}}_k$, has $|\zeta(x_k)| = r$. Then

$$\begin{aligned} I_m &= \left\langle n|(F)\prod M_{r_i,s_i}\middle|\sigma\sigma\right\rangle \frac{A\,\langle n|(F)|\sigma\sigma\rangle}{\langle n|(F)\prod M_{r_i,s_i}|\sigma\sigma\rangle} \frac{\langle n|(F)J^-(x_1)\cdots J^-(x_m)|\sigma\sigma\rangle}{\langle n|(F)|\sigma\sigma\rangle} \\ &= \left\langle n|(F)\prod M_{r_i,s_i}\middle|\sigma\sigma\right\rangle \cdot O(r^{-2nm})\,. \end{aligned} \tag{5.19}$$

When inserting (5.15) into (5.13), the most relevant contribution therefore comes from $m = 1$ and thus is of order $O(r^{-2n})$.

5.3.1 Approximating $S_\lambda(a,b)$ in terms of twist fields

Next we want to show that to leading order in $1/n$, $S_\lambda(a,b)$ can be replaced by a product of twist fields (see (1.15)). To obtain this relation we start by defining a field $\rho_{t,n}(x)$ in the same way as (5.3), but where we subtract the one-point function with respect to $\langle n|$ instead of $\langle 0|$,

$$\rho_{t,n}(x) = tJ^-(x) - f(t)\langle J^-(x)\rangle_n \mathbf{1}\,, \qquad \text{where} \qquad \langle J^-(x)\rangle_n \equiv \frac{\langle n|J^-(x)\sigma_+(b)\sigma_-(a)|0\rangle}{\langle n|\sigma_+(b)\sigma_-(a)|0\rangle}\,. \tag{5.20}$$

Let $\mathcal{C}$ be any contour from a to b. The difference $\rho_t(x) - \rho_{t,n}(x)$ has a well-defined integral along $\mathcal{C}$ since the singular contribution from the poles at a and b cancel. Using (3.21) one finds explicitly

$$\begin{aligned}\int_{\mathcal{C}}\Big(\rho_t(x) - \rho_{t,n}(x)\Big)dx &= f(t)\int_{\mathcal{C}'}\frac{\zeta^{-2n}-1}{\zeta^2-1}d\zeta = f(t)\sum_{l=0}^{n-1}\frac{2}{2l+1}\\ &= f(t)\Big(\psi(n+\tfrac{1}{2}) + \gamma + 2\ln(2)\Big)\\ &= f(t)\Big(\ln(n) + 2\ln(2) + \gamma + \frac{1}{24}n^{-2} + O(n^{-4})\Big),\end{aligned} \tag{5.21}$$

where $\mathcal{C}'$ is any contour from -1 to 1 not passing through $\zeta = 0$, γ is Euler's constant and $\psi(z)$ is the digamma function, $\psi(z) = \Gamma'(z)/\Gamma(z)$. We can then consider the product

$$\mathcal{O}_n(a,b) = \exp\left(\frac{1}{2\pi}\int_{\mathcal{C}_1+\mathcal{C}_2}\Big(\rho_1(x) - \rho_{1,n}(x)\Big)dx\right)S_1(a,b)\,, \tag{5.22}$$

which amounts to replacing $\rho_1(x)$ in the definition (5.8) of $S_1(a,b)$ by $\rho_{1,n}(x)$. When using $\rho_{t,n}$ instead of ρ_t, both the J^--integrals and the integrals of the one-point functions $\langle J^-(x)\rangle_n$ are suppressed away from a and b for large-n. It is then plausible that in the expansion of the exponential (5.8), all terms involving $\rho_{t,n}$-integrals are of order $O(n^{-1})$. This can be checked explicitly for the first few terms in the expansion, but we have no general proof. Let us hence assume

(A2) The operator $\mathcal{O}_n(a,b)$ in (5.22) has the large-n behaviour

$$\langle n|(\text{fields})\mathcal{O}_n(a,b)|0\rangle = C(a,b)\langle n|(\text{fields})\sigma_+(b)\sigma_-(a)|0\rangle\left(1 + O\left(n^{-1}\right)\right). \tag{5.23}$$

In order to obtain a $1/n$-expansion, one should thus not consider the correlator $\langle n|$ (fields) $S_1(a,b)\ |0\rangle$ directly, but rather the normalised version $\langle n|(\text{fields})\mathcal{O}_n(a,b)|0\rangle/\langle n|(\text{fields})|\sigma\sigma\rangle$. However, it is not true that the expansion of the exponential (5.8) with $\rho_{t,n}$ instead of ρ_t produces the $1/n$-expansion term by term. Instead even a term with m J^--integrations gives a contribution of order n^{-1}. This is not so surprising when one considers more carefully the regulated expression for the m'th term in the expansion. In fact, the coefficient of t^m will also contain a term of the form $f_m\int\langle J^-(x)\rangle_n dx$ from the t-expansion of $\int\rho_{t,n}(x)dx$. In this sense the subtraction scheme mixes all orders and it is only easy to extract the large-n limit, but not the subleading terms in the $1/n$-expansion.

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5.4 Fixing the normalisation of $S_\lambda(a,b)$

In section 5.1 we have partly fixed the operator $S_1(a,b)$ by requiring it to have the same monodromy as $I_1(a,b)$. Using assumption (A2) we can further fix the normalisation $C(a,b)$ by demanding $\langle n|S_1(a,b)|0\rangle/\langle n|I_1(a,b)|0\rangle = 1$ in the large-n limit. To this end, combine (5.21) and (5.22) to obtain

$$\mathcal{O}_n(a,b) = n^{\frac{f(1)}{\pi}} e^{\frac{f(1)}{\pi}(2\ln(2)+\gamma)} S_1(a,b)\left(1+O\left(n^{-1}\right)\right). \tag{5.24}$$

Next, using the above approximation together with (5.23) and (3.7) leads to

$$\langle n|S_1(a,b)|0\rangle = C(a,b) n^{-\frac{f(1)}{\pi}} e^{-\frac{f(1)}{\pi}(2\ln(2)+\gamma)} 4^{-n^2}(b-a)^{n^2-\frac{1}{8}}\left(1+O\left(n^{-1}\right)\right). \tag{5.25}$$

Comparing to the corresponding expression (4.15) for $I_1(a,b)$ then leads to the requirements

$$f(1) = \frac{\pi}{4}, \qquad C(a,b) = 2^{\frac{7}{12}}\, e^{3\zeta'(-1)+\gamma/4}(b-a)^{\frac{1}{8}}. \tag{5.26}$$

For course, $f(1)$ is in principle determined by demanding the ε-limit in (5.8) to exist, but one would need the expansion to all orders to check whether its values is indeed $\frac{\pi}{4}$. It may be taken as an encouraging sign that from $f(t) = t - \frac{1}{\pi}t^2 + \frac{1}{6}t^3 + O(t^4)$, the first three approximations to $f(1) - \frac{\pi}{4}$ are 0.21, -0.10, and 0.06.

After fixing $C(a,b)$, the operator $S_1(a,b)$ is completely determined. The definition of the regulated expression (1.11) is that $S_1(a,b)$ is given by (5.8) with $C(a,b)$ as in (5.26). In (1.11), the constant multiplying $(b-a)^{\frac{1}{8}}$ in $C(a,b)$ has been absorbed into the definition of $[\dots]_{\mathrm{reg}}$.

5.5 Singularity structure of the su(2)-currents in the presence of $S_\lambda(a,b)$

Finally we want to argue that the leading singularities as the currents approach the endpoints of $S_1(a,b)$ are the same as for the case of $I_1(a,b)$. For $I_1(a,b)$ these singularities were analysed in section 4.1, and the relevant properties are summarised in the statement (S). Here we want to present two pieces of evidence that the conditions of the statement (S) are also met for $S_1(a,b)$ in place of $I_1(a,b)$. In particular we want to argue that the leading singularity as $J^-(z)$ approaches the endpoints of $S_1(a,b)$ is a simple pole. We start by investigating the behaviour of $S_1(a,b)$ under global conformal transformations.

Let $\varphi(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$ be a Möbius transformation and U_φ be the operator implementing that transformation on the space of states (an explicit expression in terms of Virasoro generators can for example be found in [24, section 3.1]). Denote the field $\rho_t(z)$ introduced in (5.3) by $\rho_t(z;a,b)$ to keep track of the values for a and b entering its definition. Using $U_\varphi J^-(z) U_{\varphi^{-1}} = \varphi'(z) J^-(\varphi(z))$ and $U_\varphi \sigma_\pm(z) U_{\varphi^{-1}} = \varphi'(z)^{\frac{1}{16}}\sigma_\pm(\varphi(z))$ it is easy to check that

$$U_\varphi \rho_t(z;a,b) U_{\varphi^{-1}} = \varphi'(z)\rho_t\big(\varphi(z);\varphi(a),\varphi(b)\big). \tag{5.27}$$

Next we observe that for $C(a,b)$ defined in (5.26), we have $C(a,b)(\varphi'(a)\varphi'(b))^{\frac{1}{16}} = C(\varphi(a),\varphi(b))$. Thus the total effect of the Möbius transformation on $S_1(a,b)$ is

$$\begin{aligned} U_\varphi S_1(a,b) U_{\varphi^{-1}} &= C(a,b)\left(\lim_{\varepsilon\to 0} U_\varphi \exp\left(-\frac{1}{2\pi}\int_{\mathcal{C}_1^\varepsilon+\mathcal{C}_2^\varepsilon} \rho_t(x;a,b)dx\right)\sigma_+(b)\sigma_-(a)U_{\varphi^{-1}}\right)_{t=1} \\ &= S_1(\varphi(a),\varphi(b)). \end{aligned} \tag{5.28}$$

We see that $S_1(a,b)$ transforms as a product $\phi(a)\phi(b)$ of two Virasoro quasi-primary fields of conformal weight zero. This observation allows us to compute the correlator $\langle 0|J^c(z)S_1(a,b)|0\rangle$, for $c \in \{+,3,-\}$, since conformal invariance fixes a three-point function up to a constant,

$$\langle 0|J^c(z)S_1(a,b)|0\rangle = (\mathrm{const})(z-a)^{-1}\,(z-b)^{-1}\,(a-b)\,. \tag{5.29}$$

In particular this shows that for the out-state $\langle 0|$, all su(2)-currents have at most a first order pole at a and b.

As a second piece of evidence, we compute the correlator $\langle n|J^\nu(z)S_1(a,b)|0\rangle$ for $\nu=\pm$ to first order in the t-expansion (that is, in the definition (5.8) of $S_1(a,b)$ we expand the exponential to first order in t before setting $t=1$). Instead of using $S_1(a,b)$ it is convenient to use $\mathcal{O}_n(a,b)$ as defined in (5.22) which differs from $S_1(a,b)$ by a constant. One finds, with $\zeta=\zeta(z)$ and $\xi=\zeta(x)$,

$$\begin{aligned}
&\langle n|J^\nu(z)\;\;\mathcal{O}_n(a,b)|0\rangle = \qquad\qquad (5.30)\\
&\quad = C(a,b)\left(\langle n|J^\nu(z)\left(1-\frac{t}{2\pi}\int_{\mathcal{C}_1+\mathcal{C}_2}\left(J^-(x)-\langle J^-(x)\rangle_n\right)dx+O(t^2)\right)|\sigma\sigma\rangle\right)_{t=1}\\
&\quad = C(a,b)\langle n|J^\nu(z)|\sigma\sigma\rangle\left(1+\frac{t}{2\pi}(\zeta^{2\nu}-1)\int_{\mathcal{C}_1'+\mathcal{C}_2'}\frac{\xi^{-2n}}{(\xi\zeta^\nu-1)^2}d\xi+O(t^2)\right)_{t=1}.
\end{aligned}$$

Here $\mathcal{C}_1'$ is a contour from -1 to 1 passing along the upper half of the unit circle, while $\mathcal{C}_2'$ passes along the lower half of the unit circle. The integral in (5.30) is given by

$$\int_{\mathcal{C}'}\frac{x^{-2n}}{(xy-1)^2}\,dx = \left[-\frac{1}{2n+1}\frac{x^{-2n+1}}{(xy-1)^2}\,{}_2F_1\left(1,2;2n+2;(1-xy)^{-1}\right)\right]_{x=-1}^{x=1}, \tag{5.31}$$

where $\mathcal{C}'$ is a contour from -1 to 1, which also determines the relevant branch of the hypergeometric function. We are interested in the asymptotics of (5.30) for $\zeta\to\pm1$. This amounts to taking the argument of the hypergeometric function in (5.31) to infinity. The hypergeometric function has the asymptotics, for $u\to\infty$,

$${}_2F_1(1,2;2n{+}2;u) = -(2n{+}1)\,u^{-1}+O\left(u^{-2}\ln(u)\right)\,. \tag{5.32}$$

Altogether we find that the leading singularities of the integral (5.31) are first order poles at $y=\pm1$, which in (5.30) get cancelled by the prefactor $\zeta^{2\nu}{-}1$. Hence to first order in t, the leading singularity of (5.30) for $z\to a,b$ is that of $\langle n|J^\nu(z)|\sigma\sigma\rangle$, which clearly satisfies the conditions of (S).

After presenting these two calculations regarding the poles of $J^c(z)S_1(a,b)$ for $z\to a,b$, we will assume that in general

(A3) for $x=a,b$, inside any correlator, $\lim_{z\to x}(z-x)J^-(z)S_1(a,b)$ is finite, and there exists some $N>0$ such that $\lim_{z\to x}(z-x)^N J^{+,3}(z)S_1(a,b)$ is zero.

It then follows that the fields at the endpoints of $S_1(a,b)$, just as for $I_1(a,b)$, are highest weight with respect to the single valued combinations (4.4), and that in particular they are Virasoro primary of weight zero, i.e. $L_{m;a}S_1(a,b)=0=L_{m;b}S_1(a,b)$ for $m\geq 0$. Together with the fact that $I_\lambda(a,b)$ and $S_\lambda(a,b)$ have the same monodromy properties, this is very good evidence for the equality of $I_\lambda(a,b)$ and $S_\lambda(a,b)$.

6. Outlook

In this paper we have proposed and provided evidence for the operator identity $I_\lambda(a,b) = S_\lambda(a,b)$. Here $I_\lambda(a,b)$ has a simple formulation as an exponentiated integral of J^+-currents and is directly related to eigenvalue integrals in matrix models. The expression for $S_\lambda(a,b)$ is more complicated, involving twist fields and a regulator. However, properties of the large-n limit of correlators are easily understood in terms of $S_\lambda(a,b)$ while they are harder to see when using $I_\lambda(a,b)$.

There are several directions in which one can attempt to generalise the analysis of this paper.

(i) From the conformal field theory point of view, the most obvious question is whether there is a generalisation to su(2) at level $k > 1$. In this case the operator $I_\lambda(a,b)$ is still given by (1.1). However, there are now $k+1$ $\mathbb{Z}_2$-twisted highest weight representations, and there is no longer a unique (up to scalar multiples) out-state $\langle n|$ which is highest weight with respect to J^3_m and has J^3-charge n, but a finite-dimensional subspace. Both effects make it more difficult to identify the analogue of $S_\lambda(a,b)$. Independently, it is also of interest to see if correlators of $I_\lambda(a,b)$ in su$(2)_k$ do have a matrix model interpretation.

(ii) Recently a conformal field theory approach to the non-linear σ model with the complex torus $(C^*)^d$ as target space has been investigated in [19]. It uses a non-unitary CFT which contains a su(2) algebra at level $k = 0$. Similar to (1.1) it has an integrated exponential operator, which does not require normal ordering and generates a logarithmic branch cut. The above model is related to the A-model by a deformation and to the B-model by an additional T-duality. It would be very interesting to understand whether this CFT approach to the non-linear σ-model is linked to a matrix model description. On non-compact Calabi-Yau 3-folds associated to ADE singularities [15] the topological B-model, a subsector of the non-linear σ-model, has already been related to ADE-quiver matrix models [14, 15, 7], see also (iv) below.

(iii) From the matrix model point of view, one should use more general potentials than the infinite well potential that we considered here. This amounts to taking the out-state $\langle n|e^{-H}$ instead of just $\langle n|$. Also, it would be good to remove the effect of the hard edges, possibly by considering $S_\lambda(a - \varepsilon_n, b + \varepsilon_n)$ to shift the endpoints by an n-dependent amount away from the location of a cut.

(iv) There is a whole class of multi-matrix models, called ADE-quiver matrix models, which can be rewritten in terms of free bosons, or, more precisely, in terms of an ADE-WZW model at level one [31, 29, 42, 32, 15, 7]. For these one would define a number of operators $I^i_\lambda(a,b)$, indexed by simple roots α_i, $i = 1,\dots,r$. Since each of the corresponding raising operators E^{α_i} lies in an su(2)-subalgebra, one would expect that the analysis of this paper can be repeated without much modification. The comparison of perturbative versus non-perturbative moduli spaces of FZZT-branes (as mentioned in the introduction) has also been carried out for $(p,1)$-minimal string theory using a two-matrix model [28] (however, not an ADE-quiver model) and it would be interesting to compare results.

(v) Insertions of several operators $I_{\lambda_1}(a_1, b_1)$, $I_{\lambda_2}(a_2, b_2)$, ... correspond to several cuts in the complex plane. On the matrix model side one obtains in this way not a partition function with fixed filling fractions, but rather a generating function in the parameters λ_k for the various filling fractions. It would be interesting to see if the methods of this paper can be extended to be a useful tool in investigating the $1/n$ expansion of such multi-cut solutions.

We hope to address some of these points in the future.

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A. Appendix

A.1 Calculation of J^3 and J^- pole with $I_\lambda(a, b)$

In this appendix we show that the J^3- and J^--conditions of statement (S) in section 4.1 are met. For the case of J^- we need to prove that

$$\lim_{z \to b} (z - b) \left\langle \prod_j J^{a_j}(w_j)\, J^-(z) I_\lambda(a, b) \right\rangle = \text{finite}\,, \tag{A.1}$$

where the $w_j \neq a, b$ are pairwise disjoint. Then

$$\begin{aligned} &\left\langle \prod_j J^{a_j}(w_j) J^-(z) I_\lambda(a, b) \right\rangle = \\ &\quad = \frac{1}{n!} \left(\frac{\lambda}{2\pi}\right)^n \int_a^b dz_1 \cdots \int_a^b dz_n \left\langle \prod_j J^{a_j}(w_j)\, J^-(z) J^+(z_1) \cdots J^+(z_n) \right\rangle \end{aligned} \tag{A.2}$$

for some suitable n. The amplitude in the integrand can be calculated recursively, using the singular part of the operator product expansion (see for example [24]). Starting with the J^+ fields, it is then obvious that the terms that could be singular in the limit $z \to b$ arise in two ways: either we have the double pole

$$J^+(z_i) J^-(z) \sim \frac{1}{(z_i - z)^2} \tag{A.3}$$

that gives rise, after integration of z_i, to a simple pole in $(z-b)$; this contribution is therefore finite in the limit of (A.1). The second contribution comes from the simple pole

$$J^+(z_i)J^-(z) \sim \frac{2J^3(z)}{z_i - z}\,. \tag{A.4}$$

The other $J^+(z_j)$ fields can then either contract with the $J^{a_j}(w_j)$ fields, or we can get a further contraction of $J^+(z_j)$ with $J^3(z)$, which then leads to

$$J^+(z_j)J^+(z_i)J^-(z) \sim -\frac{2J^+(z)}{(z-z_i)(z-z_j)}\,. \tag{A.5}$$

In either case, it is straightforward to determine the z_i integrals, and we obtain either a $\log(z-b)$ or a $\log^2(z-b)$ term. In the limit of (A.1) these contributions therefore vanish.

The analysis for the case of J^3 is essentially identical; in this case, only the second type of terms appear, and we find with the same arguments as above that

$$\lim_{z\to b}(z-b)\left\langle \prod_j J^{a_j}(w_j)\, J^3(z) I_\lambda(a,b)\right\rangle = 0\,. \tag{A.6}$$

A.2 Existence of the regulator to order t^3

In this section we compute the first few orders in t of the function $f(t) = f_1 t + f_2 t^2 + f_3 t^3 + O(t^4)$ which enters the definition (5.5) of $S_1(a,b)$ via (5.3). To this end we make use of the decomposition (5.5) of $S_1(a,b)$. The operators $S^{\rm pert}_{\pm,\Lambda}$ are well defined if $U^{\varepsilon,\Lambda}_{\pm,t}$ (as given in (5.4)) has a finite $\varepsilon\to 0$ limit. This requirement fixes the constants f_1, f_2, f_3 uniquely. Here we will only treat $S^{\rm pert}_{+,\Lambda}(b)$ explicitly. The calculation for $S^{\rm pert}_{-,\Lambda}(a)$ is analogous and leads to the same answer.

In section 5.5 it was shown that under global conformal transformations, $S_1(a,b)$ behaves as a product $\phi(a)\phi(b)$ of Virasoro-primary fields of weight zero. We will use this freedom to assume that $b=0$ and $a=-\infty$. The question whether $S^{\rm pert}_{+,\Lambda}(b)$ is well-defined now amounts to checking that

$$U^{\varepsilon,\Lambda}_{+,t}(0)|0\rangle = \exp\left(\frac{1}{\pi}\int_\varepsilon^\Lambda \rho_t(x)dx\right)|\sigma_+\rangle \tag{A.7}$$

has a finite $\varepsilon\to 0$ limit, order by order in t (the sign difference with respect to (5.4) is due to the change of direction in the integral). Here, $\rho_t(x)$ takes the form

$$\rho_t(x) = tJ^-(x) - f(t)\lim_{a\to-\infty}\frac{\langle 0|J^-(x)\sigma_+(0)\sigma_-(a)\rangle}{\langle 0|\sigma_+(0)\sigma_-(a)\rangle} = tJ^-(x) - \frac{f(t)}{4x}\,. \tag{A.8}$$

Set further $R(x) = J^-(x) - f_1/(4x)$. Then the first few orders in the t-expansion of (A.7) read

$$\begin{aligned} U^{\varepsilon,\Lambda}_{+,t}(0)|0\rangle \;&= |\sigma_+\rangle + t\frac{1}{\pi}\int R(x)dx\,|\sigma_+\rangle + t^2\left(\frac{1}{\pi^2}\int_{x>y} R(x)R(y)dxdy - \frac{1}{\pi}\int\frac{f_2}{4x}dx\right)|\sigma_+\rangle + \\ &\quad + t^3\left(\frac{1}{\pi^3}\int_{x>y>z} R(x)R(y)R(z)\,dx\,dy\,dz - \right.\end{aligned}$$

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$$- \frac{1}{\pi^2} \int_{x>y} R(x) \frac{f_2}{4y} \, dx \, dy -$$

$$- \frac{1}{\pi^2} \int_{x>y} \frac{f_2}{4x} R(y) \, dx \, dy - \frac{1}{\pi} \int \frac{f_3}{4x} \, dx \Big) |\sigma_+\rangle + O(t^4) \,, \tag{A.9}$$

where the integrations are from ε to Λ, subject to the path ordering constraints as indicated.

Order t

Our strategy will be to expand $R(x)$ in modes around zero and analyse which modes give divergent contributions to the integral. To do so, first express $R(x)$ in the K-basis (2.19),

$$R(x) = K^3(x) - \frac{1}{2}\Big(K^+(x) - K^-(x)\Big) - \frac{f_1}{4x} \,. \tag{A.10}$$

The field $K^3(x)$ has integer modes and $K^\pm(x)$ half-integer modes. We decompose

$$\begin{aligned} K^3(x) &= K^3_{\geq}(x) + K^3_{<}(x)\,, \quad K^3_{\geq}(x) = \sum_{m \in \mathbb{Z}_{\geq 0}} x^{-m-1} K^3_m \,, \quad K^3_{<}(x) = \sum_{m \in \mathbb{Z}_{<0}} x^{-m-1} K^3_m \,, \\ K^\pm(x) &= K^\pm_{>}(x) + K^\pm_{<}(x)\,, \; K^\pm_{>}(x) = \sum_{r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} x^{-r-1} K^\pm_r \,, \; K^\pm_{<}(x) = \sum_{r \in \mathbb{Z}_{<0} + \frac{1}{2}} x^{-r-1} K^\pm_r \,. \end{aligned} \tag{A.11}$$

We also set $R_<(x)$ and $R_\geq(x)$ to be be given by (A.10) with all fields replaced by their $<$-part, respectively their $\geq$- or $>$-part; the $\frac{f_1}{4x}$ term is part of $R_\geq(x)$. The coefficient of t in (A.9) becomes

$$\frac{1}{\pi} \int R(x) \, dx \, |\sigma_+\rangle = \frac{1}{\pi} \int R_<(x) \, dx \, |\sigma_+\rangle + \frac{1}{\pi} \int \Big(K^3_0 - \frac{1}{4} f_1\Big) x^{-1} dx \, |\sigma_+\rangle \,. \tag{A.12}$$

The integral over $R_<(x)$ involves only powers x^r with $r \geq -\frac{1}{2}$ and has a finite $\varepsilon \to 0$ limit. The second integral has a log-divergence for $\varepsilon \to 0$ which has to be cancelled. This is achieved by setting $f_1 = 1$. With this choice for f_1 we have

$$R_\geq(x) \, |\sigma_+\rangle = 0 \,. \tag{A.13}$$

Order t^2

For the second order computation, we need to know the commutators $[K^a_\geq(x), K^b(y)]$ for $x > y$. These can be computed from the commutation relations of the K^a_m-modes, which are just the same as those of the J^a_m-modes given in (2.14). One finds, for $\nu = \pm$,

$$\begin{aligned} \Big[K^3_\geq(x) \,, K^3(y)\Big] &= \frac{1}{2(x-y)^2} \\ \Big[K^3_\geq(x) \,, K^\nu(y)\Big] &= \frac{\nu}{x-y} K^\nu(y) \\ \Big[K^\nu_>(x) \,, K^3(y)\Big] &= -\sqrt{\frac{y}{x}} \frac{\nu}{x-y} K^\nu(y) \\ \Big[K^\nu_>(x) \,, K^{-\nu}(y)\Big] &= \frac{1}{2(x-y)^2} \left(\sqrt{\frac{y}{x}} + \sqrt{\frac{x}{y}}\right) + \sqrt{\frac{y}{x}} \frac{2\nu}{x-y} K^3(y) \,. \end{aligned} \tag{A.14}$$

For the coefficient of t^2 in (A.9) we then obtain

$$
\begin{aligned}
&\left(\frac{1}{\pi^2}\int_{x>y} R(x)R(y)\,dx\,dy - \frac{1}{\pi}\int \frac{f_2}{4x}\,dx\right)|\sigma_+\rangle = \\
&\quad\overset{1)}{=} \left(\frac{1}{\pi^2}\int_{x>y} R_<(x)R_<(y)\,dx\,dy + \frac{1}{\pi^2}\int_{x>y}[R_\geq(x),R(y)]\,dx\,dy - \frac{f_2}{4\pi}(\ln\Lambda - \ln\varepsilon)\right)|\sigma_+\rangle \\
&\quad\overset{2)}{=} \left(-\frac{1}{2\pi^2}\int_{x>y}\frac{K^+(y)+K^-(y)}{\sqrt{x}(\sqrt{x}+\sqrt{y})}\,dx\,dy - \frac{1}{4\pi^2}\int_{x>y}\frac{1}{\sqrt{xy}(\sqrt{x}+\sqrt{y})^2}\,dx\,dy + \right.\\
&\qquad\left. + \frac{f_2}{4\pi}\ln\varepsilon + (\text{finite } \varepsilon\to 0)\right)|\sigma_+\rangle \\
&\quad\overset{3)}{=} \left(\frac{1}{4\pi^2}\ln\varepsilon + \frac{f_2}{4\pi}\ln\varepsilon + (\text{finite } \varepsilon\to 0)\right)|\sigma_+\rangle\,,
\end{aligned}
\tag{A.15}
$$

where in step 1) we replaced $R(x)$ by $R_<(x)+R_\geq(x)$ and used (A.13). For step 2) note that the integral over $R_<(x)R_<(y)$ has a finite $\varepsilon\to 0$ limit. The commutator $[R_\geq(x),R(y)]$ has been evaluated with the help of (A.14). For step 3), the singular contributions to the resulting integrals have to be extracted. The most singular term in the first integrand (expand $K^\nu(y)$ in modes) is $(\sqrt{xy}(\sqrt{x}+\sqrt{y}))^{-1} = 4\partial_x\partial_y((\sqrt{x}+\sqrt{y})\ln(\sqrt{x}+\sqrt{y}))$, so that the integral is regular for $\varepsilon\to 0$. The integrand of the second integral is $-4\partial_x\partial_y\ln(\sqrt{x}+\sqrt{y})$ so that there is a $\ln\varepsilon$ singularity. In order to have a finite $\varepsilon\to 0$ limit at order t^2 we need to choose $f_2 = -\pi^{-1}$.

Order t^3

The coefficient of t^3 in (A.9) can be written in the form $A_1+A_2+A_3$ with

$$
\begin{aligned}
A_1 &= \frac{1}{\pi^3}\int_{x>y>z} R(x)R(y)R(z)\,dx\,dy\,dz\,|\sigma_+\rangle \\
A_2 &= -\frac{f_2}{4\pi^2}\int_{x>y}\left(\frac{R(x)}{y}+\frac{R(y)}{x}\right)dx\,dy\,|\sigma_+\rangle \\
A_3 &= -\frac{f_3}{4\pi}\int\frac{1}{x}\,dx\,|\sigma_+\rangle\,.
\end{aligned}
\tag{A.16}
$$

The singular contributions to A_2 and A_3 are easy to evaluate,

$$
A_2 = \ln(\varepsilon)\frac{f_2}{4\pi^2}\int R_<(x)\,dx\,|\sigma_+\rangle + (\text{finite } \varepsilon\to 0)\,,\qquad A_3 = \ln(\varepsilon)\frac{f_3}{4\pi}|\sigma_+\rangle + (\text{finite } \varepsilon\to 0)\,.
\tag{A.17}
$$

Extracting the singular part of A_1 is some work. One first uses the commutators (A.14) to remove all positive mode parts of the fields K^a. One obtains a sum of terms, with each term being a function in x,y,z multiplying a state of the form, for $u,v\in x,y,z$,

$$
|\sigma_+\rangle\,,\qquad K^a_<(u)|\sigma_+\rangle\,,\qquad K^a_<(u)K^b_<(v)|\sigma_+\rangle\,,\qquad K^a_<(u)K^b_<(v)K^c_<(u)|\sigma_+\rangle\,.
\tag{A.18}
$$

It turns out that only the coefficients of the first two terms lead to singular behaviour as $\varepsilon\to 0$. In fact on finds

$$
A_1 = \frac{1}{4\pi^3}I|\sigma_+\rangle - \frac{1}{8\pi^3}\int_\varepsilon^\Lambda\int_\varepsilon^\Lambda\int_\varepsilon^\Lambda\frac{R_<(x)|\sigma_+\rangle}{\sqrt{yz}(\sqrt{y}+\sqrt{z})^2}dx\,dy\,dz + (\text{finite } \varepsilon\to 0)\,,
\tag{A.19}
$$

where

$$I = \int_{x>y>z} \frac{dx\,dy\,dz}{\sqrt{xyz}(\sqrt{x}+\sqrt{y})(\sqrt{x}+\sqrt{z})(\sqrt{y}+\sqrt{z})} = -\frac{\pi^2}{6}\ln(\varepsilon) + (\text{finite } \varepsilon\to 0)\,. \quad \text{(A.20)}$$

Altogether

$$\begin{aligned} A_1 + A_2 + A_3 = \ln(\varepsilon)\left(-\frac{1}{24\pi} + \frac{1}{4\pi^3}\int_\varepsilon^\Lambda R_<(x)\,dx + \frac{f_2}{4\pi^2}\int_\varepsilon^\Lambda R_<(x)\,dx + \frac{f_3}{4\pi}\right)|\sigma_+\rangle + \\ +(\text{finite } \varepsilon\to 0)\,, \end{aligned} \quad \text{(A.21)}$$

which has a finite limit iff $f_2 = -\pi^{-1}$ and $f_3 = \frac{1}{6}$. Note that the required value for f_2 agrees with the one obtained in the order t^2 computation.

References

[1] V.S. Adamchik, *Contributions to the theory of the Barnes function*, math.ca/0308086.

[2] T. Banks, M.R. Douglas, N. Seiberg and S.H. Shenker, *Microscopic and macroscopic loops in nonperturbative two-dimensional gravity*, *Phys. Lett.* **B 238** (1990) 279.

[3] M.V. Berry, *Stokes' phenomenon; smoothing a victorian discontinuity*, *Publ. Math. de IHES* **68** (1989) 211.

[4] P. Bleher and A. Its, *Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and the universality in the matrix model*, *Ann. Math.* **150** (1999) 185.

[5] J. de Boer, M.B. Halpern and N.A. Obers, *The operator algebra and twisted KZ equations of WZW orbifolds*, *JHEP* **10** (2001) 011 [hep-th/0105305].

[6] A. Borodin and P. Forrester, *Increasing subsequences and the hard-to-soft edge transition in matrix ensembles*, *J. Phys.* **A 36** (2003) 2963 [math-ph/0205007].

[7] S. Chiantese, A. Klemm and I. Runkel, *Higher order loop equations for A_r and D_r quiver matrix models*, *JHEP* **03** (2004) 033 [hep-th/0311258].

[8] P. Deift, *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, AMS 1998.

[9] P. Deift, T. Kriecherbauer, K. McLaughlin, S. Venakides and X. Zhou, *Uniform asymptotics of polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory*, *Commun. Pure Appl. Math.* **52** (1999) 1335.

[10] P. Deift and X. Zhou, *A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation*, *Ann. Math.* **137** (1993) 295.

[11] P. Di Francesco, P.H. Ginsparg and J. Zinn-Justin, *2-D gravity and random matrices*, *Phys. Rept.* **254** (1995) 1 [hep-th/9306153].

[12] P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory*, Springer 1997.

[13] R. Dijkgraaf, A. Sinkovics and M. Temurhan, *Matrix models and gravitational corrections*, *Adv. Theor. Math. Phys.* **7** (2004) 1155 [hep-th/0211241].

[14] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings and supersymmetric gauge theories*, *Nucl. Phys.* **B 644** (2002) 3 [hep-th/0206255].

[15] R. Dijkgraaf and C. Vafa, *On geometry and matrix models*, *Nucl. Phys.* **B 644** (2002) 21 [`hep-th/0207106`].

[16] R. Dijkgraaf, H.L. Verlinde and E.P. Verlinde, *Loop equations and Virasoro constraints in nonperturbative 2-D quantum gravity*, *Nucl. Phys.* **B 348** (1991) 435.

[17] L.J. Dixon, D. Friedan, E.J. Martinec and S.H. Shenker, *The conformal field theory of orbifolds*, *Nucl. Phys.* **B 282** (1987) 13.

[18] B. Eynard, *Loop equations for the semiclassical 2-matrix model with hard edges*, `math-ph/0504002`.

[19] E. Frenkel and A. Losev, *Mirror symmetry in two steps: A-I-B*, `hep-th/0505131`.

[20] M. Fukuma, H. Kawai and R. Nakayama, *Continuum Schwinger-Dyson equations and universal structures in two-dimensional quantum gravity*, *Int. J. Mod. Phys.* **A 6** (1991) 1385.

[21] M. Fukuma and S. Yahikozawa, *Nonperturbative effects in noncritical strings with soliton backgrounds*, *Phys. Lett.* **B 396** (1997) 97 [`hep-th/9609210`].

[22] Y.V. Fyodorov, *Introduction to the random matrix theory: gaussian unitary ensemble and beyond*, `math-ph/0412017`.

[23] M.R. Gaberdiel, *Fusion of twisted representations*, *Int. J. Mod. Phys.* **A 12** (1997) 5183 [`hep-th/9607036`].

[24] M.R. Gaberdiel, *An introduction to conformal field theory*, *Rept. Prog. Phys.* **63** (2000) 607 [`hep-th/9910156`].

[25] M.R. Gaberdiel and P. Goddard, *Axiomatic conformal field theory*, *Commun. Math. Phys.* **209** (2000) 549 [`hep-th/9810019`].

[26] P. Goddard and D.I. Olive, *Kac-Moody and Virasoro algebras in relation to quantum physics*, *Int. J. Mod. Phys.* **A 1** (1986) 303.

[27] I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series, and products*, Academic Press 2000, 6th ed.

[28] A. Hashimoto, M.-x. Huang, A. Klemm and D. Shih, *Open/closed string duality for topological gravity with matter*, *JHEP* **05** (2005) 007 [`hep-th/0501141`].

[29] S. Kharchev, A. Marshakov, A. Mironov, A. Morozov and S. Pakuliak, *Conformal matrix models as an alternative to conventional multimatrix models*, *Nucl. Phys.* **B 404** (1993) 717 [`hep-th/9208044`].

[30] I.K. Kostov, *Multiloop correlators for closed strings with discrete target space*, *Phys. Lett.* **B 266** (1991) 42.

[31] I.K. Kostov, *Gauge invariant matrix model for the A-D-E closed strings*, *Phys. Lett.* **B 297** (1992) 74 [`hep-th/9208053`].

[32] I.K. Kostov, *Solvable statistical models on a random lattice*, *Nucl. Phys.* **45A** *(Proc. Suppl.)* (1996) 13 [`hep-th/9509124`].

[33] I.K. Kostov, *Conformal field theory techniques in random matrix models*, `hep-th/9907060`.

[34] J. Maldacena, G.W. Moore, N. Seiberg and D. Shih, *Exact vs. semiclassical target space of the minimal string*, *JHEP* **10** (2004) 020 [`hep-th/0408039`].

[35] M. Mariño, *Les Houches lectures on matrix models and topological strings*, `hep-th/0410165`.

[36] A. Marshakov, A. Mironov and A. Morozov, *Generalized matrix models as conformal field theories: discrete case*, *Phys. Lett.* **B 265** (1991) 99.

[37] M.L. Mehta, *Random Matrices*, Academic Press 2004, 3rd ed.

[38] G.W. Moore, *Geometry of the string equations*, *Commun. Math. Phys.* **133** (1990) 261.

[39] G.W. Moore, *Matrix models of 2-D gravity and isomonodromic deformation*, *Prog. Theor. Phys. Suppl.* **102** (1990) 255.

[40] G.W. Moore, N. Seiberg and M. Staudacher, *From loops to states in 2-D quantum gravity*, *Nucl. Phys.* **B 362** (1991) 665.

[41] G.W. Moore and N. Seiberg, *From loops to fields in 2-D quantum gravity*, *Int. J. Mod. Phys.* **A 7** (1992) 2601.

[42] A. Morozov, *Integrability and matrix models*, *Phys. Usp.* **37** (1994) 1 [hep-th/9303139].

[43] N. Seiberg and D. Shih, *Minimal string theory*, *Comptes Rendus Physique* **6** (2005) 165 [hep-th/0409306].

[44] Al.B. Zamolodchikov, *Two-dimensional conformal symmetry and critical four-spin correlation functions in the Ashkin-Teller model*, *Zh. Eksp. Teor. Fiz.* **90** (1986) 1808 [*Sov. Phys. JETP* **63** (1986) 1061].

[45] Al.B. Zamolodchikov, *Conformal scalar field on the hyperelliptic curve and critical Ashkin-Teller multipoint correlation functions*, *Nucl. Phys.* **B 285** (1987) 481.